12 | SOURCES OF MAGNETIC FIELDS

Figure 12.1 An external hard drive attached to a computer works by magnetically encoding information that can be stored or retrieved quickly. A key idea in the development of digital devices is the ability to produce and use magnetic fields in this way. (credit: modification of work by “Miss Karen”/Flickr)

Chapter Outline

Introduction

In the preceding chapter, we saw that a moving charged particle produces a magnetic field. This connection between electricity and magnetism is exploited in electromagnetic devices, such as a computer hard drive. In fact, it is the underlying principle behind most of the technology in modern society, including telephones, television, computers, and the internet.

In this chapter, we examine how magnetic fields are created by arbitrary distributions of electric current, using the Biot-Savart law. Then we look at how current-carrying wires create magnetic fields and deduce the forces that arise between two current-carrying wires due to these magnetic fields. We also study the torques produced by the magnetic fields of current loops. We then generalize these results to an important law of electromagnetism, called Ampère’s law.

We examine some devices that produce magnetic fields from currents in geometries based on loops, known as solenoids and toroids. Finally, we look at how materials behave in magnetic fields and categorize materials based on their responses to magnetic fields.

12.1 | The Biot-Savart Law

Learning Objectives

By the end of this section, you will be able to:

- Explain how to derive a magnetic field from an arbitrary current in a line segment
- Calculate magnetic field from the Biot-Savart law in specific geometries, such as a current in a line and a current in a circular arc

We have seen that mass produces a gravitational field and also interacts with that field. Charge produces an electric field and also interacts with that field. Since moving charge (that is, current) interacts with a magnetic field, we might expect that it also creates that field—and it does.

The equation used to calculate the magnetic field produced by a current is known as the Biot-Savart law. It is an empirical law named in honor of two scientists who investigated the interaction between a straight, current-carrying wire and a permanent magnet. This law enables us to calculate the magnitude and direction of the magnetic field produced by a current in a wire. The **Biot-Savart law** states that at any point *P* (**Figure 12.2**), the magnetic field $d\vec{\mathbf{B}}$ due to an element $d\vec{\mathbf{l}}$ of a current-carrying wire is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}. \qquad (12.1)$$

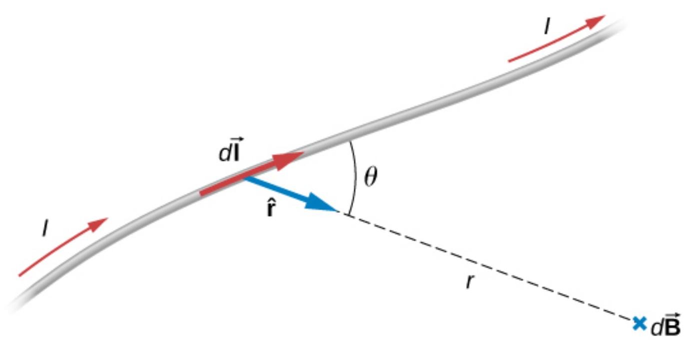


Figure 12.2 A current element $Id\vec{\mathbf{l}}$ produces a magnetic field at point *P* given by the Biot-Savart law.

The constant μ_0 is known as the **permeability of free space** and is exactly

$$\mu_0 = 4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A} \qquad (12.2)$$

in the SI system. The infinitesimal wire segment $d\vec{\mathbf{l}}$ is in the same direction as the current *I* (assumed positive), *r* is the distance from $d\vec{\mathbf{l}}$ to *P* and $\hat{\mathbf{r}}$ is a unit vector that points from $d\vec{\mathbf{l}}$ to *P*, as shown in the figure.

The direction of $d\vec{\mathbf{B}}$ is determined by applying the right-hand rule to the vector product $d\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. The magnitude of $d\vec{\mathbf{B}}$ is

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} \qquad (12.3)$$

where θ is the angle between $d\,\vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$. Notice that if $\theta = 0$, then $d\,\vec{\mathbf{B}} = \vec{\mathbf{0}}$. The field produced by a current element $Id\,\vec{\mathbf{l}}$ has no component parallel to $d\,\vec{\mathbf{l}}$.

The magnetic field due to a finite length of current-carrying wire is found by integrating **Equation 12.3** along the wire, giving us the usual form of the Biot-Savart law.

Biot-Savart law

The magnetic field $\vec{\mathbf{B}}$ due to an element $d\,\vec{\mathbf{l}}$ of a current-carrying wire is given by

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I\,d\,\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}. \tag{12.4}$$

Since this is a vector integral, contributions from different current elements may not point in the same direction. Consequently, the integral is often difficult to evaluate, even for fairly simple geometries. The following strategy may be helpful.

Problem-Solving Strategy: Solving Biot-Savart Problems

To solve Biot-Savart law problems, the following steps are helpful:

1. Identify that the Biot-Savart law is the chosen method to solve the given problem. If there is symmetry in the problem comparing $\vec{\mathbf{B}}$ and $\mathbf{d}\,\vec{\mathbf{l}}$, Ampère's law may be the preferred method to solve the question.
2. Draw the current element length $d\,\vec{\mathbf{l}}$ and the unit vector $\hat{\mathbf{r}}$, noting that $d\,\vec{\mathbf{l}}$ points in the direction of the current and $\hat{\mathbf{r}}$ points from the current element toward the point where the field is desired.
3. Calculate the cross product $d\,\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field according to the Biot-Savart law.
4. Use **Equation 12.4** and substitute all given quantities into the expression to solve for the magnetic field. Note all variables that remain constant over the entire length of the wire may be factored out of the integration.
5. Use the right-hand rule to verify the direction of the magnetic field produced from the current or to write down the direction of the magnetic field if only the magnitude was solved for in the previous part.

Example 12.1

Calculating Magnetic Fields of Short Current Segments

A short wire of length 1.0 cm carries a current of 2.0 A in the vertical direction (**Figure 12.3**). The rest of the wire is shielded so it does not add to the magnetic field produced by the wire. Calculate the magnetic field at point *P*, which is 1 meter from the wire in the *x*-direction.

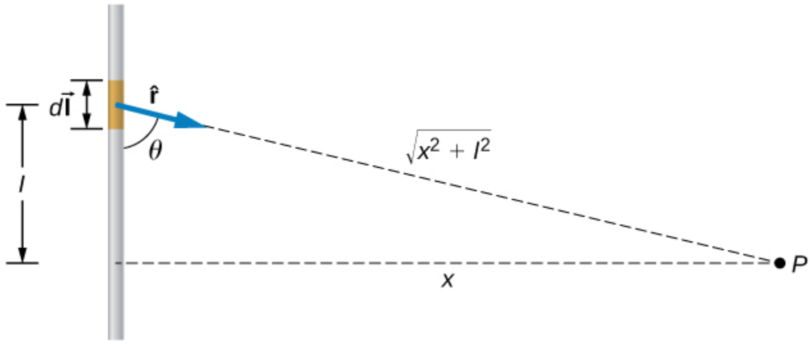


Figure 12.3 A small line segment carries a current I in the vertical direction. What is the magnetic field at a distance x from the segment?

Strategy

We can determine the magnetic field at point P using the Biot-Savart law. Since the current segment is much smaller than the distance x, we can drop the integral from the expression. The integration is converted back into a summation, but only for small dl, which we now write as Δl. Another way to think about it is that each of the radius values is nearly the same, no matter where the current element is on the line segment, if Δl is small compared to x. The angle θ is calculated using a tangent function. Using the numbers given, we can calculate the magnetic field at P.

Solution

The angle between $\Delta \vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$ is calculated from trigonometry, knowing the distances l and x from the problem:

$$\theta = \tan^{-1}\left(\frac{1 \text{ m}}{0.01 \text{ m}}\right) = 89.4°.$$

The magnetic field at point P is calculated by the Biot-Savart law:

$$B = \frac{\mu_0}{4\pi}\frac{I\Delta l \sin\theta}{r^2} = (1 \times 10^{-7}\,\text{T} \cdot \text{m/A})\left(\frac{2\,\text{A}(0.01\text{ m})\sin(89.4°)}{(1\text{ m})^2}\right) = 2.0 \times 10^{-9}\,\text{T}.$$

From the right-hand rule and the Biot-Savart law, the field is directed into the page.

Significance

This approximation is only good if the length of the line segment is very small compared to the distance from the current element to the point. If not, the integral form of the Biot-Savart law must be used over the entire line segment to calculate the magnetic field.

12.1 Check Your Understanding Using **Example 12.1**, at what distance would P have to be to measure a magnetic field half of the given answer?

Example 12.2

Calculating Magnetic Field of a Circular Arc of Wire

A wire carries a current I in a circular arc with radius R swept through an arbitrary angle θ (**Figure 12.4**). Calculate the magnetic field at the center of this arc at point P.

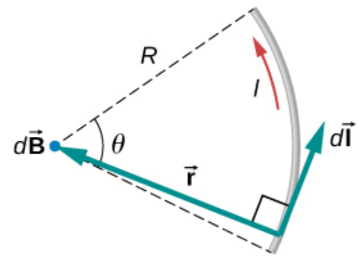


Figure 12.4 A wire segment carrying a current I. The path $d\vec{\mathbf{l}}$ and radial direction $\hat{\mathbf{r}}$ are indicated.

Strategy

We can determine the magnetic field at point P using the Biot-Savart law. The radial and path length directions are always at a right angle, so the cross product turns into multiplication. We also know that the distance along the path dl is related to the radius times the angle θ (in radians). Then we can pull all constants out of the integration and solve for the magnetic field.

Solution

The Biot-Savart law starts with the following equation:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}.$$

As we integrate along the arc, all the contributions to the magnetic field are in the same direction (out of the page), so we can work with the magnitude of the field. The cross product turns into multiplication because the path dl and the radial direction are perpendicular. We can also substitute the arc length formula, $dl = rd\theta$:

$$B = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{Ir\, d\theta}{r^2}.$$

The current and radius can be pulled out of the integral because they are the same regardless of where we are on the path. This leaves only the integral over the angle,

$$B = \frac{\mu_0 I}{4\pi r} \int_{\text{wire}} d\theta.$$

The angle varies on the wire from 0 to θ ; hence, the result is

$$B = \frac{\mu_0 I\theta}{4\pi r}.$$

Significance

The direction of the magnetic field at point P is determined by the right-hand rule, as shown in the previous chapter. If there are other wires in the diagram along with the arc, and you are asked to find the net magnetic field, find each contribution from a wire or arc and add the results by superposition of vectors. Make sure to pay attention to the direction of each contribution. Also note that in a symmetric situation, like a straight or circular wire, contributions from opposite sides of point P cancel each other.

12.2 Check Your Understanding The wire loop forms a full circle of radius R and current I. What is the magnitude of the magnetic field at the center?

12.2 | Magnetic Field Due to a Thin Straight Wire

Learning Objectives

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a thin, straight wire.
- Determine the dependence of the magnetic field from a thin, straight wire based on the distance from it and the current flowing in the wire.
- Sketch the magnetic field created from a thin, straight wire by using the second right-hand rule.

How much current is needed to produce a significant magnetic field, perhaps as strong as Earth's field? Surveyors will tell you that overhead electric power lines create magnetic fields that interfere with their compass readings. Indeed, when Oersted discovered in 1820 that a current in a wire affected a compass needle, he was not dealing with extremely large currents. How does the shape of wires carrying current affect the shape of the magnetic field created? We noted in Chapter 28 that a current loop created a magnetic field similar to that of a bar magnet, but what about a straight wire? We can use the Biot-Savart law to answer all of these questions, including determining the magnetic field of a long straight wire.

Figure 12.5 shows a section of an infinitely long, straight wire that carries a current *I*. What is the magnetic field at a point *P*, located a distance *R* from the wire?

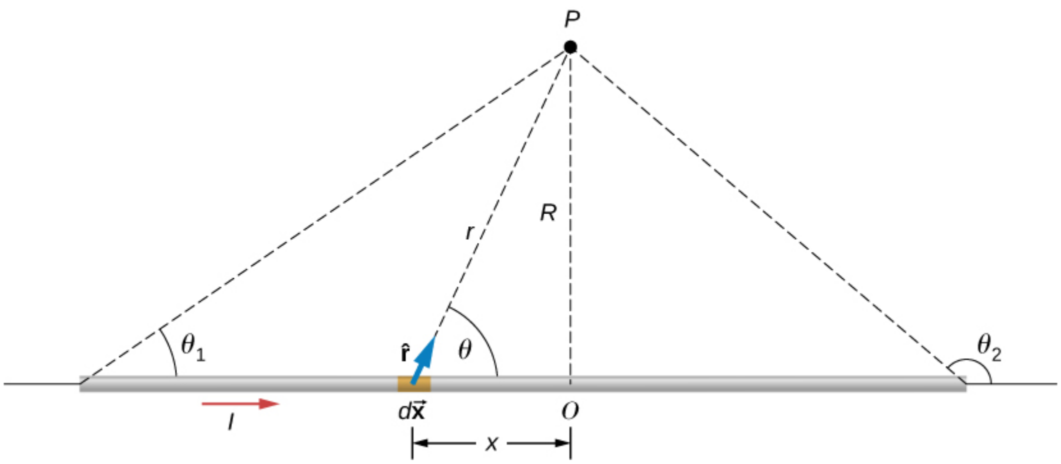


Figure 12.5 A section of a thin, straight current-carrying wire. The independent variable θ has the limits θ_1 and θ_2.

Let's begin by considering the magnetic field due to the current element $I\, d\vec{\mathbf{x}}$ located at the position *x*. Using the right-hand rule 1 from the previous chapter, $d\vec{\mathbf{x}} \times \hat{\mathbf{r}}$ points out of the page for any element along the wire. At point *P*, therefore, the magnetic fields due to all current elements have the same direction. This means that we can calculate the net field there by evaluating the scalar sum of the contributions of the elements. With $\left|d\vec{\mathbf{x}} \times \hat{\mathbf{r}}\right| = (dx)(1)\sin\theta,$ we have from the Biot-Savart law

$$B = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I \sin\theta\, dx}{r^2}. \tag{12.5}$$

The wire is symmetrical about point *O*, so we can set the limits of the integration from zero to infinity and double the answer, rather than integrate from negative infinity to positive infinity. Based on the picture and geometry, we can write expressions for *r* and $\sin\theta$ in terms of *x* and *R*, namely:

$$\begin{aligned} r &= \sqrt{x^2 + R^2} \\ \sin\theta &= \frac{R}{\sqrt{x^2 + R^2}}. \end{aligned}$$

Substituting these expressions into **Equation 12.5**, the magnetic field integration becomes

$$B = \frac{\mu_0 I}{2\pi} \int_0^{\infty} \frac{R\,dx}{(x^2 + R^2)^{3/2}}. \tag{12.6}$$

Evaluating the integral yields

$$B = \frac{\mu_0 I}{2\pi R} \left[\frac{x}{(x^2 + R^2)^{1/2}} \right]_0^{\infty}. \tag{12.7}$$

Substituting the limits gives us the solution

$$B = \frac{\mu_0 I}{2\pi R}. \tag{12.8}$$

The magnetic field lines of the infinite wire are circular and centered at the wire (**Figure 12.6**), and they are identical in every plane perpendicular to the wire. Since the field decreases with distance from the wire, the spacing of the field lines must increase correspondingly with distance. The direction of this magnetic field may be found with a second form of the right-hand rule (illustrated in **Figure 12.6**). If you hold the wire with your right hand so that your thumb points along the current, then your fingers wrap around the wire in the same sense as $\vec{\mathbf{B}}$.

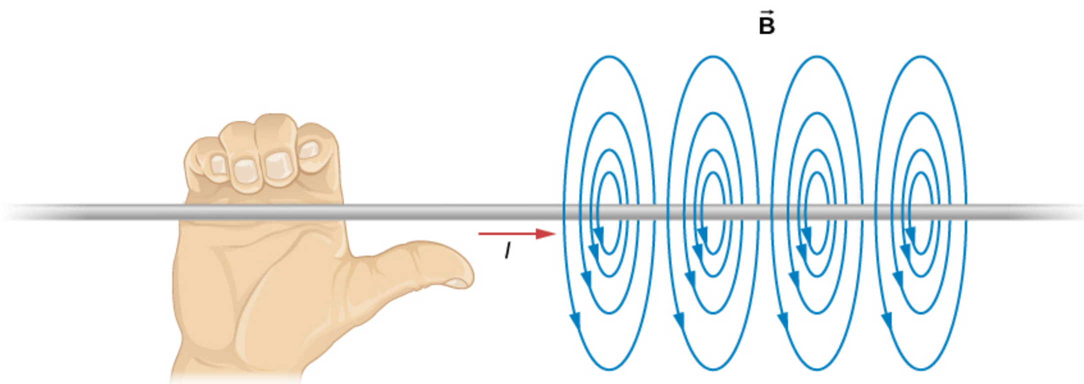


Figure 12.6 Some magnetic field lines of an infinite wire. The direction of $\vec{\mathbf{B}}$ can be found with a form of the right-hand rule.

The direction of the field lines can be observed experimentally by placing several small compass needles on a circle near the wire, as illustrated in **Figure 12.7**. When there is no current in the wire, the needles align with Earth's magnetic field. However, when a large current is sent through the wire, the compass needles all point tangent to the circle. Iron filings sprinkled on a horizontal surface also delineate the field lines, as shown in **Figure 12.7**.

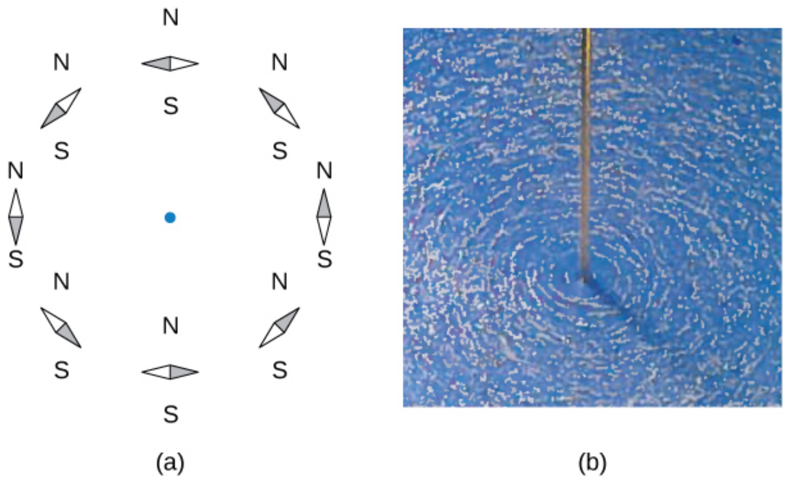


Figure 12.7 The shape of the magnetic field lines of a long wire can be seen using (a) small compass needles and (b) iron filings.

Example 12.3

Calculating Magnetic Field Due to Three Wires

Three wires sit at the corners of a square, all carrying currents of 2 amps into the page as shown in **Figure 12.8**. Calculate the magnitude of the magnetic field at the other corner of the square, point *P*, if the length of each side of the square is 1 cm.

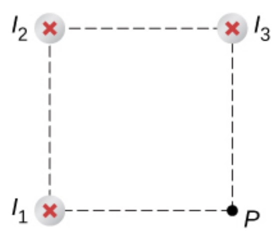


Figure 12.8 Three wires have current flowing into the page. The magnetic field is determined at the fourth corner of the square.

Strategy

The magnetic field due to each wire at the desired point is calculated. The diagonal distance is calculated using the Pythagorean theorem. Next, the direction of each magnetic field's contribution is determined by drawing a circle centered at the point of the wire and out toward the desired point. The direction of the magnetic field contribution from that wire is tangential to the curve. Lastly, working with these vectors, the resultant is calculated.

Solution

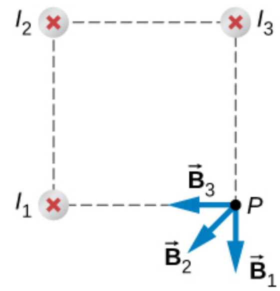
Wires 1 and 3 both have the same magnitude of magnetic field contribution at point *P*:

$$B_1 = B_3 = \frac{\mu_o I}{2\pi R} = \frac{(4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A})(2\,\text{A})}{2\pi(0.01\,\text{m})} = 4 \times 10^{-5}\,\text{T}.$$

Wire 2 has a longer distance and a magnetic field contribution at point *P* of:

$$B_2 = \frac{\mu_o I}{2\pi R} = \frac{(4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A})(2\,\text{A})}{2\pi(0.01414\,\text{m})} = 3 \times 10^{-5}\,\text{T}.$$

The vectors for each of these magnetic field contributions are shown.



The magnetic field in the *x*-direction has contributions from wire 3 and the *x*-component of wire 2:

$$B_{\text{net } x} = -4 \times 10^{-5}\,\text{T} - 2.83 \times 10^{-5}\,\text{T}\cos(45°) = -6 \times 10^{-5}\,\text{T}.$$

The *y*-component is similarly the contributions from wire 1 and the *y*-component of wire 2:

$$B_{\text{net } y} = -4 \times 10^{-5}\,\text{T} - 2.83 \times 10^{-5}\,\text{T}\sin(45°) = -6 \times 10^{-5}\,\text{T}.$$

Therefore, the net magnetic field is the resultant of these two components:

$$B_{\text{net}} = \sqrt{B^2_{\text{net } x} + B^2_{\text{net } y}}$$
$$B_{\text{net}} = \sqrt{(-6 \times 10^{-5}\,\text{T})^2 + (-6 \times 10^{-5}\,\text{T})^2}$$
$$B_{\text{net}} = 8 \times 10^{-5}\,\text{T}.$$

Significance

The geometry in this problem results in the magnetic field contributions in the *x*- and *y*-directions having the same magnitude. This is not necessarily the case if the currents were different values or if the wires were located in different positions. Regardless of the numerical results, working on the components of the vectors will yield the resulting magnetic field at the point in need.

12.3 Check Your Understanding Using **Example 12.3**, keeping the currents the same in wires 1 and 3, what should the current be in wire 2 to counteract the magnetic fields from wires 1 and 3 so that there is no net magnetic field at point P?

12.3 | Magnetic Force between Two Parallel Currents

Learning Objectives

By the end of this section, you will be able to:

- Explain how parallel wires carrying currents can attract or repel each other
- Define the ampere and describe how it is related to current-carrying wires
- Calculate the force of attraction or repulsion between two current-carrying wires

You might expect that two current-carrying wires generate significant forces between them, since ordinary currents produce magnetic fields and these fields exert significant forces on ordinary currents. But you might not expect that the force between wires is used to define the ampere. It might also surprise you to learn that this force has something to do with why large circuit breakers burn up when they attempt to interrupt large currents.

The force between two long, straight, and parallel conductors separated by a distance *r* can be found by applying what we have developed in the preceding sections. **Figure 12.9** shows the wires, their currents, the field created by one wire, and the consequent force the other wire experiences from the created field. Let us consider the field produced by wire 1 and the force it exerts on wire 2 (call the force F_2). The field due to I_1 at a distance *r* is

$$B_1 = \frac{\mu_0 I_1}{2\pi r} \quad (12.9)$$

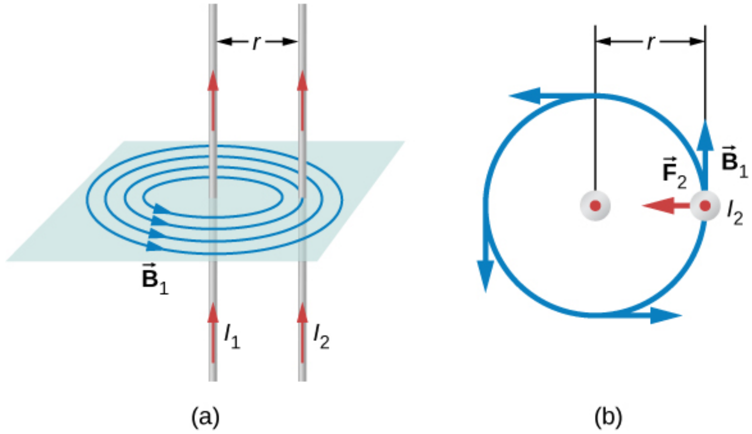


Figure 12.9 (a) The magnetic field produced by a long straight conductor is perpendicular to a parallel conductor, as indicated by right-hand rule (RHR)-2. (b) A view from above of the two wires shown in (a), with one magnetic field line shown for wire 1. RHR-1 shows that the force between the parallel conductors is attractive when the currents are in the same direction. A similar analysis shows that the force is repulsive between currents in opposite directions.

This field is uniform from the wire 1 and perpendicular to it, so the force F_2 it exerts on a length *l* of wire 2 is given by $F = IlB\sin\theta$ with $\sin\theta = 1$:

$$F_2 = I_2 l B_1. \quad (12.10)$$

The forces on the wires are equal in magnitude, so we just write *F* for the magnitude of F_2. (Note that $\vec{\mathbf{F}}_1 = -\vec{\mathbf{F}}_2$.) Since the wires are very long, it is convenient to think in terms of *F/l*, the force per unit length. Substituting the expression for B_1 into **Equation 12.10** and rearranging terms gives

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}. \quad (12.11)$$

The ratio *F/l* is the force per unit length between two parallel currents I_1 and I_2 separated by a distance *r*. The force is attractive if the currents are in the same direction and repulsive if they are in opposite directions.

This force is responsible for the *pinch effect* in electric arcs and other plasmas. The force exists whether the currents are in wires or not. It is only apparent if the overall charge density is zero; otherwise, the Coulomb repulsion overwhelms the magnetic attraction. In an electric arc, where charges are moving parallel to one another, an attractive force squeezes currents into a smaller tube. In large circuit breakers, such as those used in neighborhood power distribution systems, the pinch effect can concentrate an arc between plates of a switch trying to break a large current, burn holes, and even ignite the equipment. Another example of the pinch effect is found in the solar plasma, where jets of ionized material, such as solar flares, are shaped by magnetic forces.

The definition of the ampere is based on the force between current-carrying wires. Note that for long, parallel wires separated by 1 meter with each carrying 1 ampere, the force per meter is

$$\frac{F}{l} = \frac{\left(4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A}\right)(1\ \text{A})^2}{(2\pi)(1\ \text{m})} = 2 \times 10^{-7}\ \text{N/m}. \quad (12.12)$$

Since μ_0 is exactly $4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ by definition, and because $1\ \text{T} = 1\ \text{N/(A} \cdot \text{m)},$ the force per meter is exactly

2×10^{-7} N/m. This is the basis of the definition of the ampere.

Infinite-length wires are impractical, so in practice, a current balance is constructed with coils of wire separated by a few centimeters. Force is measured to determine current. This also provides us with a method for measuring the coulomb. We measure the charge that flows for a current of one ampere in one second. That is, $1\text{ C} = 1\text{ A} \cdot \text{s}$. For both the ampere and the coulomb, the method of measuring force between conductors is the most accurate in practice.

Example 12.4

Calculating Forces on Wires

Two wires, both carrying current out of the page, have a current of magnitude 5.0 mA. The first wire is located at (0.0 cm, 3.0 cm) while the other wire is located at (4.0 cm, 0.0 cm) as shown in **Figure 12.10**. What is the magnetic force per unit length of the first wire on the second and the second wire on the first?

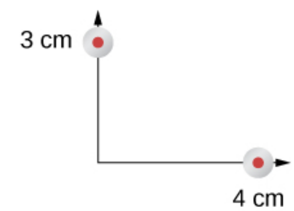


Figure 12.10 Two current-carrying wires at given locations with currents out of the page.

Strategy

Each wire produces a magnetic field felt by the other wire. The distance along the hypotenuse of the triangle between the wires is the radial distance used in the calculation to determine the force per unit length. Since both wires have currents flowing in the same direction, the direction of the force is toward each other.

Solution

The distance between the wires results from finding the hypotenuse of a triangle:

$$r = \sqrt{(3.0\text{ cm})^2 + (4.0\text{ cm})^2} = 5.0\text{ cm}.$$

The force per unit length can then be calculated using the known currents in the wires:

$$\frac{F}{l} = \frac{\left(4\pi \times 10^{-7}\text{ T} \cdot \text{m/A}\right)\left(5 \times 10^{-3}\text{ A}\right)^2}{(2\pi)(5 \times 10^{-2}\text{ m})} = 1 \times 10^{-10}\text{ N/m}.$$

The force from the first wire pulls the second wire. The angle between the radius and the *x*-axis is

$$\theta = \tan^{-1}\left(\frac{3\text{ cm}}{4\text{ cm}}\right) = 36.9°.$$

The unit vector for this is calculated by

$$\cos(36.9°)\,\hat{\mathbf{i}} - \sin(36.9°)\,\hat{\mathbf{j}} = 0.8\,\hat{\mathbf{i}} - 0.6\,\hat{\mathbf{j}}.$$

Therefore, the force per unit length from wire one on wire 2 is

$$\frac{\vec{\mathbf{F}}}{l} = (1 \times 10^{-10}\text{ N/m}) \times (0.8\,\hat{\mathbf{i}} - 0.6\,\hat{\mathbf{j}}) = (8 \times 10^{-11}\,\hat{\mathbf{i}} - 6 \times 10^{-11}\,\hat{\mathbf{j}})\text{ N/m}.$$

The force per unit length from wire 2 on wire 1 is the negative of the previous answer:

$$\frac{\vec{\mathbf{F}}}{l} = (-8 \times 10^{-11}\,\hat{\mathbf{i}} + 6 \times 10^{-11}\,\hat{\mathbf{j}})\text{N/m}.$$

Significance

These wires produced magnetic fields of equal magnitude but opposite directions at each other's locations. Whether the fields are identical or not, the forces that the wires exert on each other are always equal in magnitude and opposite in direction (Newton's third law).

12.4 Check Your Understanding Two wires, both carrying current out of the page, have a current of magnitude 2.0 mA and 3.0 mA, respectively. The first wire is located at (0.0 cm, 5.0 cm) while the other wire is located at (12.0 cm, 0.0 cm). What is the magnitude of the magnetic force per unit length of the first wire on the second and the second wire on the first?

12.4 | Magnetic Field of a Current Loop

Learning Objectives

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a current in a loop of wire at a point along a line perpendicular to thep lane of the loop.
- Determine the magnetic field of an arc of current.

The circular loop of **Figure 12.11** has a radius R, carries a current I, and lies in the xz-plane. What is the magnetic field due to the current at an arbitrary point P along the axis of the loop?

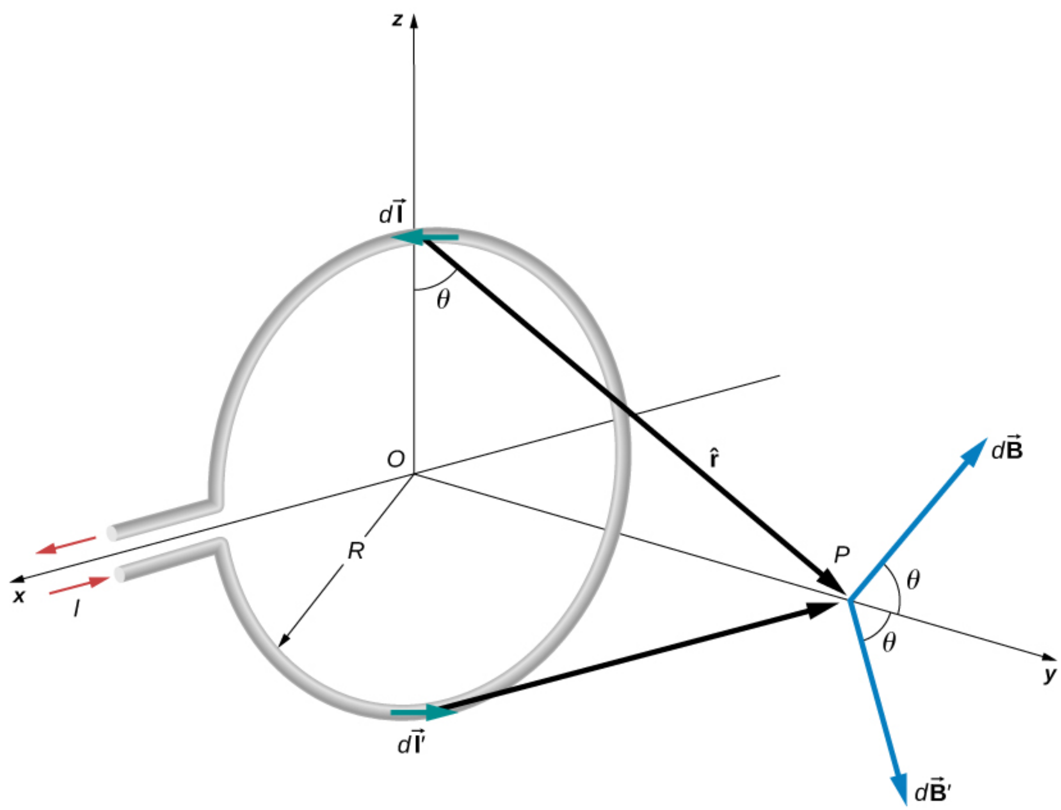


Figure 12.11 Determining the magnetic field at point P along the axis of a current-carrying loop of wire.

We can use the Biot-Savart law to find the magnetic field due to a current. We first consider arbitrary segments on opposite sides of the loop to qualitatively show by the vector results that the net magnetic field direction is along the central axis from the loop. From there, we can use the Biot-Savart law to derive the expression for magnetic field.

Let P be a distance y from the center of the loop. From the right-hand rule, the magnetic field $d\vec{\mathbf{B}}$ at P, produced by the current element $I\,d\vec{\mathbf{l}}$, is directed at an angle θ above the y-axis as shown. Since $d\vec{\mathbf{l}}$ is parallel along the x-axis and $\hat{\mathbf{r}}$ is in the yz-plane, the two vectors are perpendicular, so we have

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,dl}{y^2 + R^2} \tag{12.13}$$

where we have used $r^2 = y^2 + R^2$.

Now consider the magnetic field $d\vec{\mathbf{B}}'$ due to the current element $I\,d\vec{\mathbf{l}}'$, which is directly opposite $I\,d\vec{\mathbf{l}}$ on the loop. The magnitude of $d\vec{\mathbf{B}}'$ is also given by **Equation 12.13**, but it is directed at an angle θ below the y-axis. The components of $d\vec{\mathbf{B}}$ and $d\vec{\mathbf{B}}'$ perpendicular to the y-axis therefore cancel, and in calculating the net magnetic field, only the components along the y-axis need to be considered. The components perpendicular to the axis of the loop sum to zero in pairs. Hence at point P:

$$\vec{\mathbf{B}} = \hat{\mathbf{j}}\int_{\text{loop}} dB\cos\theta = \hat{\mathbf{j}}\frac{\mu_0 I}{4\pi}\int_{\text{loop}}\frac{\cos\theta\,dl}{y^2 + R^2}. \tag{12.14}$$

For all elements $d\vec{\mathbf{l}}$ on the wire, y, R, and $\cos\theta$ are constant and are related by

$$\cos\theta = \frac{R}{\sqrt{y^2 + R^2}}.$$

Now from **Equation 12.14**, the magnetic field at P is

$$\vec{\mathbf{B}} = \hat{\mathbf{j}}\frac{\mu_0 IR}{4\pi(y^2 + R^2)^{3/2}}\int_{\text{loop}} dl = \frac{\mu_0 IR^2}{2(y^2 + R^2)^{3/2}}\hat{\mathbf{j}} \tag{12.15}$$

where we have used $\int_{\text{loop}} dl = 2\pi R.$ As discussed in the previous chapter, the closed current loop is a magnetic dipole of moment $\vec{\mu} = IA\hat{\mathbf{n}}$. For this example, $A = \pi R^2$ and $\hat{\mathbf{n}} = \hat{\mathbf{j}}$, so the magnetic field at P can also be written as

$$\vec{\mathbf{B}} = \frac{\mu_0 \mu\hat{\mathbf{j}}}{2\pi(y^2 + R^2)^{3/2}}. \tag{12.16}$$

By setting $y = 0$ in **Equation 12.16**, we obtain the magnetic field at the center of the loop:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R}\hat{\mathbf{j}}. \tag{12.17}$$

This equation becomes $B = \mu_0 nI/(2R)$ for a flat coil of n loops per length. It can also be expressed as

$$\vec{\mathbf{B}} = \frac{\mu_0\vec{\mu}}{2\pi R^3}. \tag{12.18}$$

If we consider $y \gg R$ in **Equation 12.16**, the expression reduces to an expression known as the magnetic field from a dipole:

$$\vec{\mathbf{B}} = \frac{\mu_0 \vec{\mu}}{2\pi y^3}. \tag{12.19}$$

The calculation of the magnetic field due to the circular current loop at points off-axis requires rather complex mathematics, so we'll just look at the results. The magnetic field lines are shaped as shown in **Figure 12.12**. Notice that one field line follows the axis of the loop. This is the field line we just found. Also, very close to the wire, the field lines are almost circular, like the lines of a long straight wire.

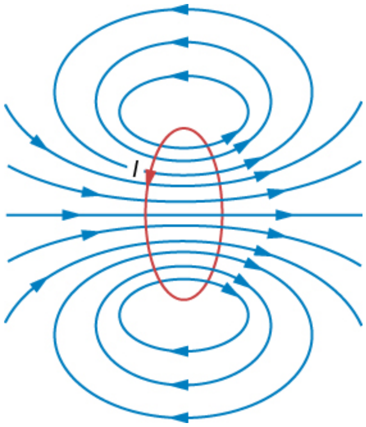


Figure 12.12 Sketch of the magnetic field lines of a circular current loop.

Example 12.5

Magnetic Field between Two Loops

Two loops of wire carry the same current of 10 mA, but flow in opposite directions as seen in **Figure 12.13**. One loop is measured to have a radius of $R = 50\,\text{cm}$ while the other loop has a radius of $2R = 100\,\text{cm}$. The distance from the first loop to the point where the magnetic field is measured is 0.25 m, and the distance from that point to the second loop is 0.75 m. What is the magnitude of the net magnetic field at point *P*?

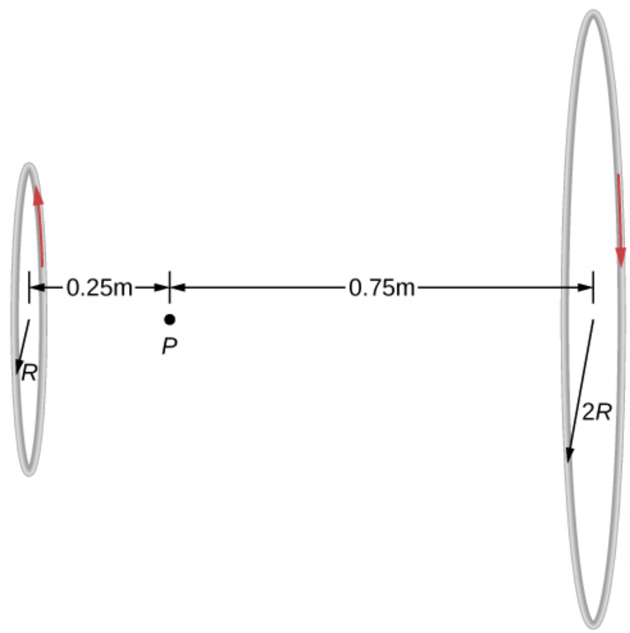


Figure 12.13 Two loops of different radii have the same current but flowing in opposite directions. The magnetic field at point *P* is measured to be zero.

Strategy

The magnetic field at point P has been determined in **Equation 12.15**. Since the currents are flowing in opposite directions, the net magnetic field is the difference between the two fields generated by the coils. Using the given quantities in the problem, the net magnetic field is then calculated.

Solution

Solving for the net magnetic field using **Equation 12.15** and the given quantities in the problem yields

$$\begin{aligned} B &= \frac{\mu_0 I R_1{}^2}{2\left(y_1{}^2 + R_1{}^2\right)^{3/2}} - \frac{\mu_0 I R_2{}^2}{2\left(y_2{}^2 + R_2{}^2\right)^{3/2}} \\ B &= \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(0.010\,\text{A})(0.5\,\text{m})^2}{2((0.25\,\text{m})^2 + (0.5\,\text{m})^2)^{3/2}} - \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(0.010\,\text{A})(1.0\,\text{m})^2}{2((0.75\,\text{m})^2 + (1.0\,\text{m})^2)^{3/2}} \\ B &= 5.77 \times 10^{-9}\,\text{T to the right.} \end{aligned}$$

Significance

Helmholtz coils typically have loops with equal radii with current flowing in the same direction to have a strong uniform field at the midpoint between the loops. A similar application of the magnetic field distribution created by Helmholtz coils is found in a magnetic bottle that can temporarily trap charged particles. See **Magnetic Forces and Fields** for a discussion on this.

12.5 Check Your Understanding Using **Example 12.5**, at what distance would you have to move the first coil to have zero measurable magnetic field at point P?

12.5 | Ampère's Law

Learning Objectives

By the end of this section, you will be able to:

- Explain how Ampère's law relates the magnetic field produced by a current to the value of the current
- Calculate the magnetic field from a long straight wire, either thin or thick, by Ampère's law

A fundamental property of a static magnetic field is that, unlike an electrostatic field, it is not conservative. A conservative field is one that does the same amount of work on a particle moving between two different points regardless of the path chosen. Magnetic fields do not have such a property. Instead, there is a relationship between the magnetic field and its source, electric current. It is expressed in terms of the line integral of $\vec{\mathbf{B}}$ and is known as **Ampère's law**. This law can also be derived directly from the Biot-Savart law. We now consider that derivation for the special case of an infinite, straight wire.

Figure 12.14 shows an arbitrary plane perpendicular to an infinite, straight wire whose current I is directed out of the page. The magnetic field lines are circles directed counterclockwise and centered on the wire. To begin, let's consider $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ over the closed paths M and N. Notice that one path (M) encloses the wire, whereas the other (N) does not. Since the field lines are circular, $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ is the product of B and the projection of dl onto the circle passing through $d\vec{\mathbf{l}}$. If the radius of this particular circle is r, the projection is $r d\theta$, and

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = Br\,d\theta.$$

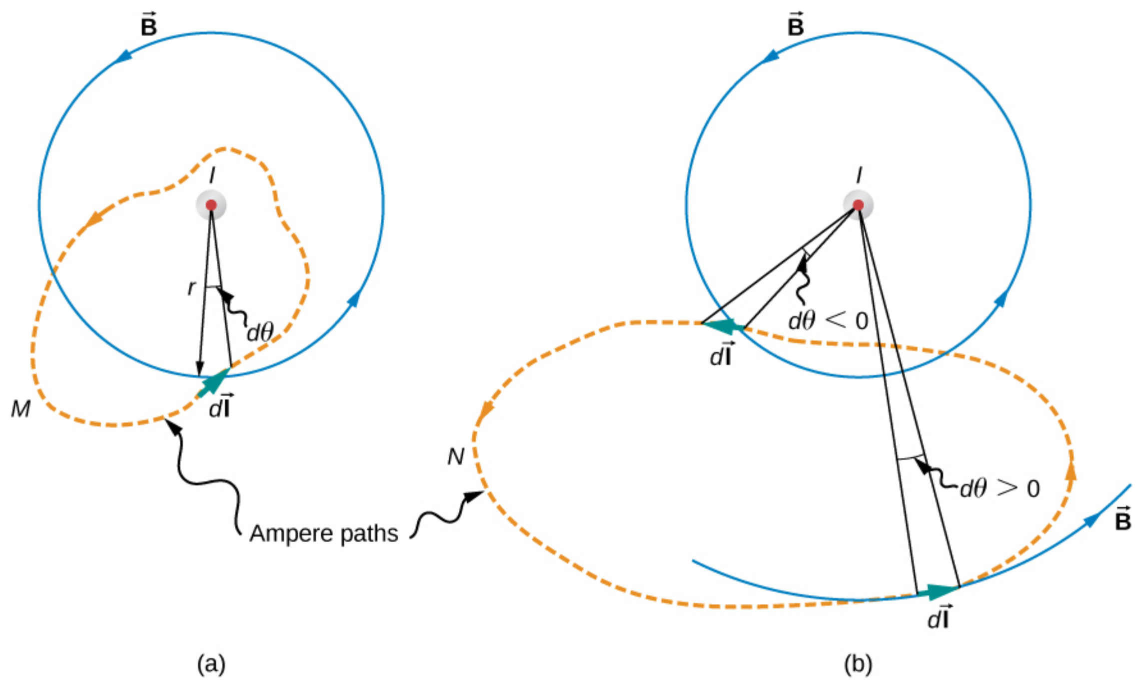


Figure 12.14 The current I of a long, straight wire is directed out of the page. The integral $\oint d\theta$ equals 2π and 0, respectively, for paths M and N.

With $\vec{\mathbf{B}}$ given by **Equation 12.9**,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint \left(\frac{\mu_0 I}{2\pi r}\right) r\, d\theta = \frac{\mu_0 I}{2\pi} \oint d\theta. \tag{12.20}$$

For path M, which circulates around the wire, $\oint_M d\theta = 2\pi$ and

$$\oint_M \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I. \tag{12.21}$$

Path N, on the other hand, circulates through both positive (counterclockwise) and negative (clockwise) $d\theta$ (see **Figure 12.14**), and since it is closed, $\oint_N d\theta = 0$. Thus for path N,

$$\oint_N \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = 0. \tag{12.22}$$

The extension of this result to the general case is Ampère's law.

Ampère's law

Over an arbitrary closed path,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I \tag{12.23}$$

where I is the total current passing through any open surface S whose perimeter is the path of integration. Only currents inside the path of integration need be considered.

To determine whether a specific current I is positive or negative, curl the fingers of your right hand in the direction of the path of integration, as shown in **Figure 12.14**. If I passes through S in the same direction as your extended thumb, I is positive; if I passes through S in the direction opposite to your extended thumb, it is negative.

Problem-Solving Strategy: Ampère's Law

To calculate the magnetic field created from current in wire(s), use the following steps:

1. Identify the symmetry of the current in the wire(s). If there is no symmetry, use the Biot-Savart law to determine the magnetic field.
2. Determine the direction of the magnetic field created by the wire(s) by right-hand rule 2.
3. Chose a path loop where the magnetic field is either constant or zero.
4. Calculate the current inside the loop.
5. Calculate the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ around the closed loop.
6. Equate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{\mathbf{B}}$.

Example 12.6

Using Ampère's Law to Calculate the Magnetic Field Due to a Wire

Use Ampère's law to calculate the magnetic field due to a steady current *I* in an infinitely long, thin, straight wire as shown in **Figure 12.15**.

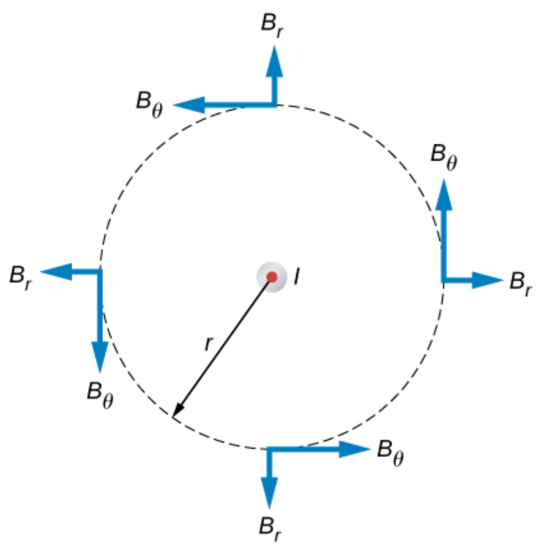


Figure 12.15 The possible components of the magnetic field *B* due to a current *I*, which is directed out of the page. The radial component is zero because the angle between the magnetic field and the path is at a right angle.

Strategy

Consider an arbitrary plane perpendicular to the wire, with the current directed out of the page. The possible magnetic field components in this plane, B_r and B_θ, are shown at arbitrary points on a circle of radius *r* centered on the wire. Since the field is cylindrically symmetric, neither B_r nor B_θ varies with the position on this circle. Also from symmetry, the radial lines, if they exist, must be directed either all inward or all outward from the wire. This means, however, that there must be a net magnetic flux across an arbitrary cylinder concentric with the wire. The radial component of the magnetic field must be zero because $\vec{\mathbf{B}}_r \cdot d\vec{\mathbf{l}} = 0$. Therefore, we can apply Ampère's law to the circular path as shown.

Solution

Over this path $\vec{\mathbf{B}}$ is constant and parallel to $d\vec{\mathbf{l}}$, so

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B_\theta \oint dl = B_\theta (2\pi r).$$

Thus Ampère’s law reduces to

$$B_\theta (2\pi r) = \mu_0 I.$$

Finally, since B_θ is the only component of $\vec{\mathbf{B}}$, we can drop the subscript and write

$$B = \frac{\mu_0 I}{2\pi r}.$$

This agrees with the Biot-Savart calculation above.

Significance

Ampère’s law works well if you have a path to integrate over which $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ has results that are easy to simplify. For the infinite wire, this works easily with a path that is circular around the wire so that the magnetic field factors out of the integration. If the path dependence looks complicated, you can always go back to the Biot-Savart law and use that to find the magnetic field.

Example 12.7

Calculating the Magnetic Field of a Thick Wire with Ampère’s Law

The radius of the long, straight wire of **Figure 12.16** is *a*, and the wire carries a current I_0 that is distributed uniformly over its cross-section. Find the magnetic field both inside and outside the wire.

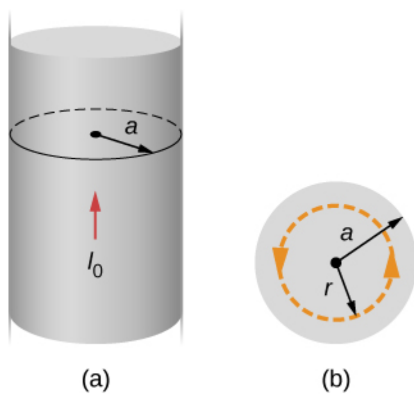


Figure 12.16 (a) A model of a current-carrying wire of radius *a* and current I_0. (b) A cross-section of the same wire showing the radius *a* and the Ampère’s loop of radius *r*.

Strategy

This problem has the same geometry as **Example 12.6**, but the enclosed current changes as we move the integration path from outside the wire to inside the wire, where it doesn’t capture the entire current enclosed (see **Figure 12.16**).

Solution

For any circular path of radius *r* that is centered on the wire,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint Bdl = B\oint dl = B(2\pi r).$$

From Ampère’s law, this equals the total current passing through any surface bounded by the path of integration.

Consider first a circular path that is inside the wire $(r \leq a)$ such as that shown in part (a) of **Figure 12.16**. We need the current I passing through the area enclosed by the path. It’s equal to the current density J times the area enclosed. Since the current is uniform, the current density inside the path equals the current density in the whole wire, which is $I_0 / \pi a^2$. Therefore the current I passing through the area enclosed by the path is

$$I = \frac{\pi r^2}{\pi a^2} I_0 = \frac{r^2}{a^2} I_0.$$

We can consider this ratio because the current density J is constant over the area of the wire. Therefore, the current density of a part of the wire is equal to the current density in the whole area. Using Ampère’s law, we obtain

$$B(2\pi r) = \mu_0 \left(\frac{r^2}{a^2}\right) I_0,$$

and the magnetic field inside the wire is

$$B = \frac{\mu_0 I_0}{2\pi} \frac{r}{a^2} \ (r \leq a).$$

Outside the wire, the situation is identical to that of the infinite thin wire of the previous example; that is,

$$B = \frac{\mu_0 I_0}{2\pi r} \ (r \geq a).$$

The variation of B with r is shown in **Figure 12.17**.

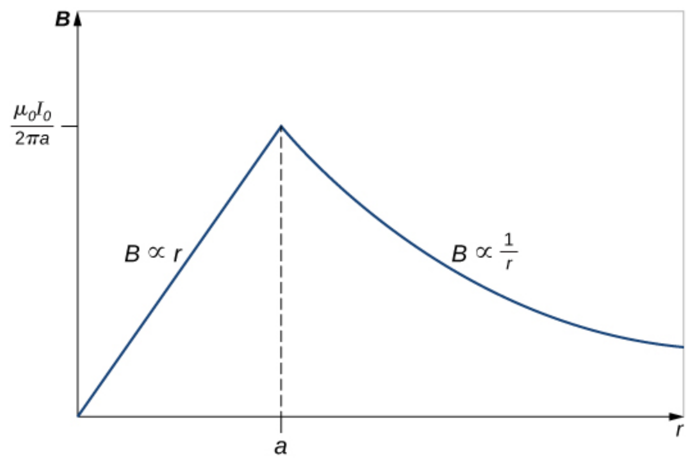


Figure 12.17 Variation of the magnetic field produced by a current I_0 in a long, straight wire of radius a.

Significance

The results show that as the radial distance increases inside the thick wire, the magnetic field increases from zero to a familiar value of the magnetic field of a thin wire. Outside the wire, the field drops off regardless of whether it was a thick or thin wire.

This result is similar to how Gauss’s law for electrical charges behaves inside a uniform charge distribution, except that Gauss’s law for electrical charges has a uniform volume distribution of charge, whereas Ampère’s law here has a uniform area of current distribution. Also, the drop-off outside the thick wire is similar to how an electric field drops off outside of a linear charge distribution, since the two cases have the same geometry and neither case depends on the configuration of charges or currents once the loop is outside the distribution.

Example 12.8

Using Ampère's Law with Arbitrary Paths

Use Ampère's law to evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the current configurations and paths in **Figure 12.18**.

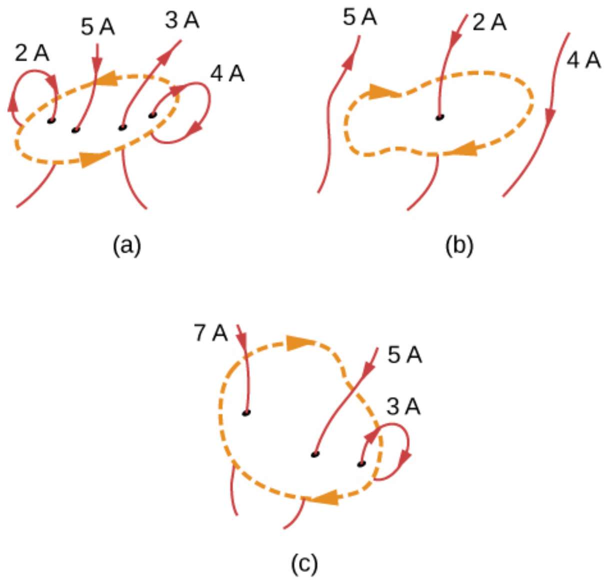


Figure 12.18 Current configurations and paths for **Example 12.8**.

Strategy

Ampère's law states that $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I$ where I is the total current passing through the enclosed loop. The quickest way to evaluate the integral is to calculate $\mu_0 I$ by finding the net current through the loop. Positive currents flow with your right-hand thumb if your fingers wrap around in the direction of the loop. This will tell us the sign of the answer.

Solution

(a) The current going downward through the loop equals the current going out of the loop, so the net current is zero. Thus, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = 0$.

(b) The only current to consider in this problem is 2A because it is the only current inside the loop. The right-hand rule shows us the current going downward through the loop is in the positive direction. Therefore, the answer is

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0(2\text{ A}) = 2.51 \times 10^{-6}\text{T} \cdot \text{m/A}.$$

(c) The right-hand rule shows us the current going downward through the loop is in the positive direction. There are $7\text{A} + 5\text{A} = 12\text{A}$ of current going downward and –3 A going upward. Therefore, the total current is 9 A and

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0(9\text{ A}) = 5.65 \times 10^{-6}\text{T} \cdot \text{m/A}.$$

Significance

If the currents all wrapped around so that the same current went into the loop and out of the loop, the net current would be zero and no magnetic field would be present. This is why wires are very close to each other in an electrical cord. The currents flowing toward a device and away from a device in a wire equal zero total current flow through an Ampère loop around these wires. Therefore, no stray magnetic fields can be present from cords carrying current.

12.6 Check Your Understanding Consider using Ampère's law to calculate the magnetic fields of a finite straight wire and of a circular loop of wire. Why is it not useful for these calculations?

12.6 | Solenoids and Toroids

Learning Objectives

By the end of this section, you will be able to:

- Establish a relationship for how the magnetic field of a solenoid varies with distance and current by using both the Biot-Savart law and Ampère's law
- Establish a relationship for how the magnetic field of a toroid varies with distance and current by using Ampère's law

Two of the most common and useful electromagnetic devices are called solenoids and toroids. In one form or another, they are part of numerous instruments, both large and small. In this section, we examine the magnetic field typical of these devices.

Solenoids

A long wire wound in the form of a helical coil is known as a **solenoid**. Solenoids are commonly used in experimental research requiring magnetic fields. A solenoid is generally easy to wind, and near its center, its magnetic field is quite uniform and directly proportional to the current in the wire.

Figure 12.19 shows a solenoid consisting of N turns of wire tightly wound over a length L. A current I is flowing along the wire of the solenoid. The number of turns per unit length is N/L; therefore, the number of turns in an infinitesimal length dy are $(N/L)dy$ turns. This produces a current

$$dI = \frac{NI}{L}dy. \tag{12.24}$$

We first calculate the magnetic field at the point P of **Figure 12.19**. This point is on the central axis of the solenoid. We are basically cutting the solenoid into thin slices that are dy thick and treating each as a current loop. Thus, dI is the current through each slice. The magnetic field $d\vec{\mathbf{B}}$ due to the current dI in dy can be found with the help of **Equation 12.15** and **Equation 12.24**:

$$d\vec{\mathbf{B}} = \frac{\mu_0 R^2 dI}{2(y^2 + R^2)^{3/2}}\hat{\mathbf{j}} = \left(\frac{\mu_0 I R^2 N}{2L}\hat{\mathbf{j}}\right)\frac{dy}{(y^2 + R^2)^{3/2}} \tag{12.25}$$

where we used **Equation 12.24** to replace dI. The resultant field at P is found by integrating $d\vec{\mathbf{B}}$ along the entire length of the solenoid. It's easiest to evaluate this integral by changing the independent variable from y to θ. From inspection of **Figure 12.19**, we have:

$$\sin\theta = \frac{y}{\sqrt{y^2 + R^2}}. \tag{12.26}$$

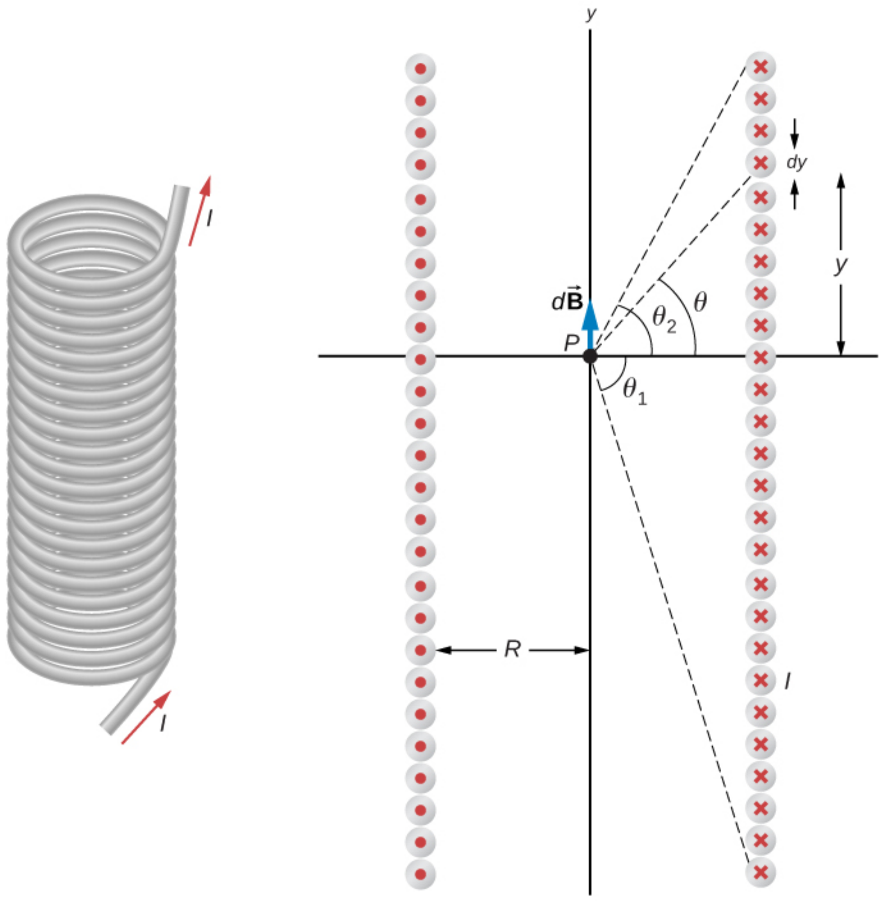


Figure 12.19 (a) A solenoid is a long wire wound in the shape of a helix. (b) The magnetic field at the point P on the axis of the solenoid is the net field due to all of the current loops.

Taking the differential of both sides of this equation, we obtain

$$\begin{aligned}\cos\theta\, d\theta &= \left[-\frac{y^2}{(y^2+R^2)^{3/2}} + \frac{1}{\sqrt{y^2+R^2}}\right]dy \\ &= \frac{R^2\, dy}{(y^2+R^2)^{3/2}}.\end{aligned}$$

When this is substituted into the equation for $d\vec{\mathbf{B}}$, we have

$$\vec{\mathbf{B}} = \frac{\mu I_0 N}{2L}\hat{\mathbf{j}}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu I_0 N}{2L}(\sin\theta_2 - \sin\theta_1)\hat{\mathbf{j}}, \tag{12.27}$$

which is the magnetic field along the central axis of a finite solenoid.

Of special interest is the infinitely long solenoid, for which $L \to \infty$. From a practical point of view, the infinite solenoid is one whose length is much larger than its radius $(L \gg R)$. In this case, $\theta_1 = \frac{-\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$. Then from **Equation 12.27**, the magnetic field along the central axis of an infinite solenoid is

$$\vec{\mathbf{B}} = \frac{\mu_0 IN}{2L}\hat{\mathbf{j}}[\sin(\pi/2) - \sin(-\pi/2)] = \frac{\mu_0 IN}{L}\hat{\mathbf{j}}$$

or

$$\vec{\mathbf{B}} = \mu_0 n I \hat{\mathbf{j}}, \tag{12.28}$$

where n is the number of turns per unit length. You can find the direction of $\vec{\mathbf{B}}$ with a right-hand rule: Curl your fingers in the direction of the current, and your thumb points along the magnetic field in the interior of the solenoid.

We now use these properties, along with Ampère's law, to calculate the magnitude of the magnetic field at any location inside the infinite solenoid. Consider the closed path of **Figure 12.20**. Along segment 1, $\vec{\mathbf{B}}$ is uniform and parallel to the path. Along segments 2 and 4, $\vec{\mathbf{B}}$ is perpendicular to part of the path and vanishes over the rest of it. Therefore, segments 2 and 4 do not contribute to the line integral in Ampère's law. Along segment 3, $\vec{\mathbf{B}} = 0$ because the magnetic field is zero outside the solenoid. If you consider an Ampère's law loop outside of the solenoid, the current flows in opposite directions on different segments of wire. Therefore, there is no enclosed current and no magnetic field according to Ampère's law. Thus, there is no contribution to the line integral from segment 3. As a result, we find

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \int_1 \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = Bl. \tag{12.29}$$

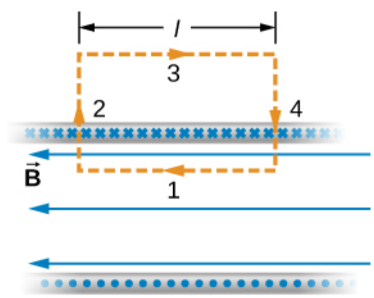


Figure 12.20 The path of integration used in Ampère's law to evaluate the magnetic field of an infinite solenoid.

The solenoid has n turns per unit length, so the current that passes through the surface enclosed by the path is nlI. Therefore, from Ampère's law,

$$Bl = \mu_0 nlI$$

and

$$B = \mu_0 nI \tag{12.30}$$

within the solenoid. This agrees with what we found earlier for B on the central axis of the solenoid. Here, however, the location of segment 1 is arbitrary, so we have found that this equation gives the magnetic field everywhere inside the infinite solenoid.

Outside the solenoid, one can draw an Ampère's law loop around the entire solenoid. This would enclose current flowing in both directions. Therefore, the net current inside the loop is zero. According to Ampère's law, if the net current is zero, the magnetic field must be zero. Therefore, for locations outside of the solenoid's radius, the magnetic field is zero.

When a patient undergoes a magnetic resonance imaging (MRI) scan, the person lies down on a table that is moved into the center of a large solenoid that can generate very large magnetic fields. The solenoid is capable of these high fields from high currents flowing through superconducting wires. The large magnetic field is used to change the spin of protons in the patient's body. The time it takes for the spins to align or relax (return to original orientation) is a signature of different tissues that can be analyzed to see if the structures of the tissues is normal (**Figure 12.21**).

Figure 12.21 In an MRI machine, a large magnetic field is generated by the cylindrical solenoid surrounding the patient. (credit: Liz West)

Example 12.9

Magnetic Field Inside a Solenoid

A solenoid has 300 turns wound around a cylinder of diameter 1.20 cm and length 14.0 cm. If the current through the coils is 0.410 A, what is the magnitude of the magnetic field inside and near the middle of the solenoid?

Strategy

We are given the number of turns and the length of the solenoid so we can find the number of turns per unit length. Therefore, the magnetic field inside and near the middle of the solenoid is given by **Equation 12.30**. Outside the solenoid, the magnetic field is zero.

Solution

The number of turns per unit length is

$$n = \frac{300 \text{ turns}}{0.140 \text{ m}} = 2.14 \times 10^3 \text{ turns/m}.$$

The magnetic field produced inside the solenoid is

$$\begin{aligned} B &= \mu_0 nI = (4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(2.14 \times 10^3 \text{ turns/m})(0.410 \text{ A}) \\ B &= 1.10 \times 10^{-3} \text{ T}. \end{aligned}$$

Significance

This solution is valid only if the length of the solenoid is reasonably large compared with its diameter. This example is a case where this is valid.

12.7 Check Your Understanding What is the ratio of the magnetic field produced from using a finite formula over the infinite approximation for an angle θ of (a) $85°$? (b) $89°$? The solenoid has 1000 turns in 50 cm with a current of 1.0 A flowing through the coils

Toroids

A toroid is a donut-shaped coil closely wound with one continuous wire, as illustrated in part (a) of **Figure 12.22**. If the toroid has *N* windings and the current in the wire is *I*, what is the magnetic field both inside and outside the toroid?

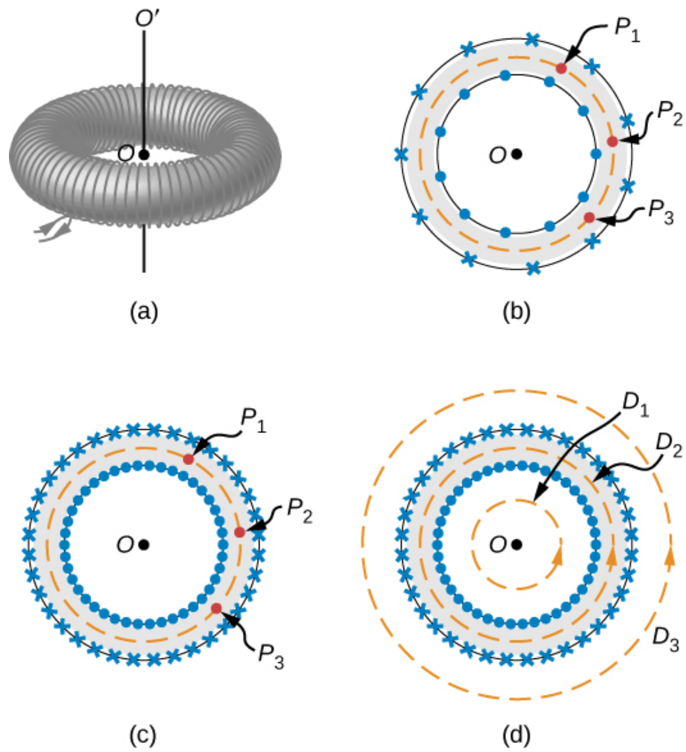


Figure 12.22 (a) A toroid is a coil wound into a donut-shaped object. (b) A loosely wound toroid does not have cylindrical symmetry. (c) In a tightly wound toroid, cylindrical symmetry is a very good approximation. (d) Several paths of integration for Ampère's law.

We begin by assuming cylindrical symmetry around the axis *OO*'. Actually, this assumption is not precisely correct, for as part (b) of **Figure 12.22** shows, the view of the toroidal coil varies from point to point (for example, P_1, P_2, and P_3) on a circular path centered around *OO*'. However, if the toroid is tightly wound, all points on the circle become essentially equivalent [part (c) of **Figure 12.22**], and cylindrical symmetry is an accurate approximation.

With this symmetry, the magnetic field must be tangent to and constant in magnitude along any circular path centered on *OO*'. This allows us to write for each of the paths D_1, D_2, and D_3 shown in part (d) of **Figure 12.22**,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r). \tag{12.31}$$

Ampère's law relates this integral to the net current passing through any surface bounded by the path of integration. For a path that is external to the toroid, either no current passes through the enclosing surface (path D_1), or the current passing through the surface in one direction is exactly balanced by the current passing through it in the opposite direction (path D_3). In either case, there is no net current passing through the surface, so

$$\oint B(2\pi r) = 0$$

and

$$B = 0 \quad \text{(outside the toroid).} \tag{12.32}$$

The turns of a toroid form a helix, rather than circular loops. As a result, there is a small field external to the coil; however, the derivation above holds if the coils were circular.

For a circular path within the toroid (path D_2), the current in the wire cuts the surface *N* times, resulting in a net current *NI* through the surface. We now find with Ampère's law,

$$B(2\pi r) = \mu_0 NI$$

and

$$B = \frac{\mu_0 NI}{2\pi r} \quad \text{(within the toroid).} \tag{12.33}$$

The magnetic field is directed in the counterclockwise direction for the windings shown. When the current in the coils is reversed, the direction of the magnetic field also reverses.

The magnetic field inside a toroid is not uniform, as it varies inversely with the distance r from the axis OO'. However, if the central radius R (the radius midway between the inner and outer radii of the toroid) is much larger than the cross-sectional diameter of the coils r, the variation is fairly small, and the magnitude of the magnetic field may be calculated by **Equation 12.33** where $r = R$.

12.7 | Magnetism in Matter

Learning Objectives

By the end of this section, you will be able to:

- Classify magnetic materials as paramagnetic, diamagnetic, or ferromagnetic, based on their response to a magnetic field
- Sketch how magnetic dipoles align with the magnetic field in each type of substance
- Define hysteresis and magnetic susceptibility, which determines the type of magnetic material

Why are certain materials magnetic and others not? And why do certain substances become magnetized by a field, whereas others are unaffected? To answer such questions, we need an understanding of magnetism on a microscopic level.

Within an atom, every electron travels in an orbit and spins on an internal axis. Both types of motion produce current loops and therefore magnetic dipoles. For a particular atom, the net magnetic dipole moment is the vector sum of the magnetic dipole moments. Values of μ for several types of atoms are given in **Table 12.1**. Notice that some atoms have a zero net dipole moment and that the magnitudes of the nonvanishing moments are typically $10^{-23}\,\text{A}\cdot\text{m}^2$.

Atom	Magnetic Moment $\left(10^{-24}\,\text{A}\cdot\text{m}^2\right)$
H	9.27
He	0
Li	9.27
O	13.9
Na	9.27
S	13.9

Table 12.1 Magnetic Moments of Some Atoms

A handful of matter has approximately 10^{26} atoms and ions, each with its magnetic dipole moment. If no external magnetic field is present, the magnetic dipoles are randomly oriented—as many are pointed up as down, as many are pointed east as west, and so on. Consequently, the net magnetic dipole moment of the sample is zero. However, if the sample is placed in a magnetic field, these dipoles tend to align with the field (see **Equation 12.14**), and this alignment determines how the sample responds to the field. On the basis of this response, a material is said to be either paramagnetic, ferromagnetic, or diamagnetic.

In a **paramagnetic material**, only a small fraction (roughly one-third) of the magnetic dipoles are aligned with the applied field. Since each dipole produces its own magnetic field, this alignment contributes an extra magnetic field, which enhances the applied field. When a **ferromagnetic material** is placed in a magnetic field, its magnetic dipoles also become aligned;

furthermore, they become locked together so that a permanent magnetization results, even when the field is turned off or reversed. This permanent magnetization happens in ferromagnetic materials but not paramagnetic materials. **Diamagnetic materials** are composed of atoms that have no net magnetic dipole moment. However, when a diamagnetic material is placed in a magnetic field, a magnetic dipole moment is directed opposite to the applied field and therefore produces a magnetic field that opposes the applied field. We now consider each type of material in greater detail.

Paramagnetic Materials

For simplicity, we assume our sample is a long, cylindrical piece that completely fills the interior of a long, tightly wound solenoid. When there is no current in the solenoid, the magnetic dipoles in the sample are randomly oriented and produce no net magnetic field. With a solenoid current, the magnetic field due to the solenoid exerts a torque on the dipoles that tends to align them with the field. In competition with the aligning torque are thermal collisions that tend to randomize the orientations of the dipoles. The relative importance of these two competing processes can be estimated by comparing the energies involved. From **Equation 12.14**, the energy difference between a magnetic dipole aligned with and against a magnetic field is $U_B = 2\mu B$. If $\mu = 9.3 \times 10^{-24}\,\text{A} \cdot \text{m}^2$ (the value of atomic hydrogen) and B = 1.0 T, then

$$U_B = 1.9 \times 10^{-23}\,\text{J}.$$

At a room temperature of $27\,°\text{C},$ the thermal energy per atom is

$$U_T \approx kT = (1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K}) = 4.1 \times 10^{-21}\,\text{J},$$

which is about 220 times greater than U_B. Clearly, energy exchanges in thermal collisions can seriously interfere with the alignment of the magnetic dipoles. As a result, only a small fraction of the dipoles is aligned at any instant.

The four sketches of **Figure 12.23** furnish a simple model of this alignment process. In part (a), before the field of the solenoid (not shown) containing the paramagnetic sample is applied, the magnetic dipoles are randomly oriented and there is no net magnetic dipole moment associated with the material. With the introduction of the field, a partial alignment of the dipoles takes place, as depicted in part (b). The component of the net magnetic dipole moment that is perpendicular to the field vanishes. We may then represent the sample by part (c), which shows a collection of magnetic dipoles completely aligned with the field. By treating these dipoles as current loops, we can picture the dipole alignment as equivalent to a current around the surface of the material, as in part (d). This fictitious surface current produces its own magnetic field, which enhances the field of the solenoid.

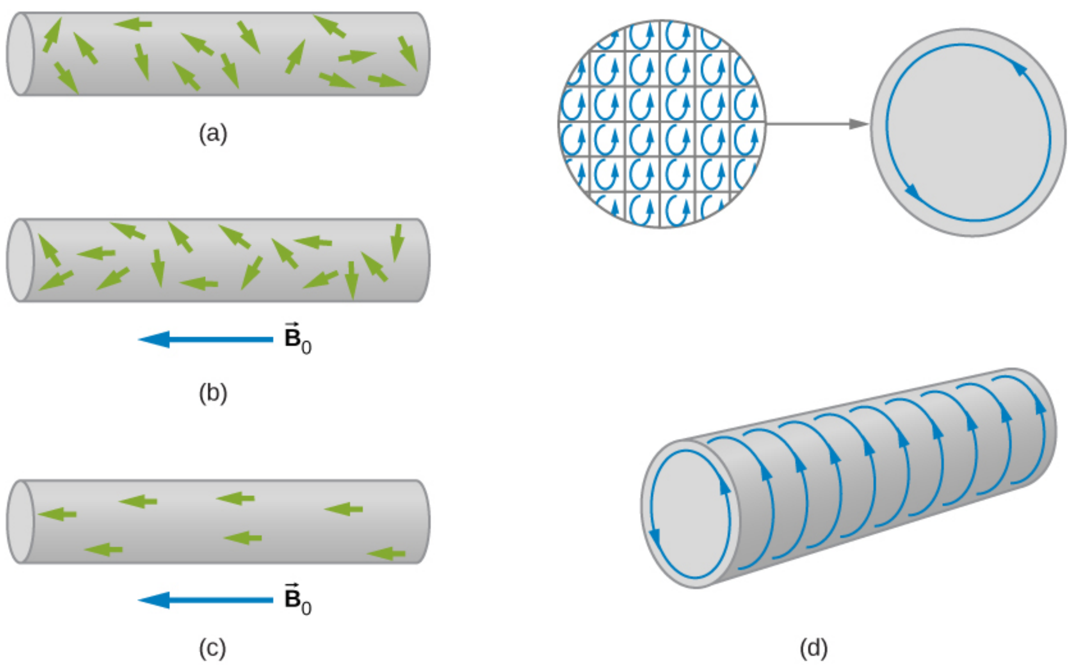


Figure 12.23 The alignment process in a paramagnetic material filling a solenoid (not shown). (a) Without an applied field, the magnetic dipoles are randomly oriented. (b) With a field, partial alignment occurs. (c) An equivalent representation of part (b). (d) The internal currents cancel, leaving an effective surface current that produces a magnetic field similar to that of a finite solenoid.

We can express the total magnetic field $\vec{\mathbf{B}}$ in the material as

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \vec{\mathbf{B}}_m, \tag{12.34}$$

where $\vec{\mathbf{B}}_0$ is the field due to the current I_0 in the solenoid and $\vec{\mathbf{B}}_m$ is the field due to the surface current I_m around the sample. Now $\vec{\mathbf{B}}_m$ is usually proportional to $\vec{\mathbf{B}}_0,$ a fact we express by

$$\vec{\mathbf{B}}_m = \chi \vec{\mathbf{B}}_0, \tag{12.35}$$

where χ is a dimensionless quantity called the **magnetic susceptibility**. Values of χ for some paramagnetic materials are given in Table 12.2. Since the alignment of magnetic dipoles is so weak, χ is very small for paramagnetic materials. By combining Equation 12.34 and Equation 12.35, we obtain:

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \chi \vec{\mathbf{B}}_0 = (1+\chi)\vec{\mathbf{B}}_0. \tag{12.36}$$

For a sample within an infinite solenoid, this becomes

$$B = (1+\chi)\mu_0 nI. \tag{12.37}$$

This expression tells us that the insertion of a paramagnetic material into a solenoid increases the field by a factor of $(1+\chi)$. However, since χ is so small, the field isn't enhanced very much.

The quantity

$$\mu = (1+\chi)\mu_0. \tag{12.38}$$

is called the magnetic permeability of a material. In terms of $\mu,$ Equation 12.37 can be written as

$$B = \mu nI \tag{12.39}$$

for the filled solenoid.

Paramagnetic Materials	χ	Diamagnetic Materials	χ
Aluminum	2.2×10^{-5}	Bismuth	-1.7×10^{-5}
Calcium	1.4×10^{-5}	Carbon (diamond)	-2.2×10^{-5}
Chromium	3.1×10^{-4}	Copper	-9.7×10^{-6}
Magnesium	1.2×10^{-5}	Lead	-1.8×10^{-5}
Oxygen gas (1 atm)	1.8×10^{-6}	Mercury	-2.8×10^{-5}
Oxygen liquid (90 K)	3.5×10^{-3}	Hydrogen gas (1 atm)	-2.2×10^{-9}
Tungsten	6.8×10^{-5}	Nitrogen gas (1 atm)	-6.7×10^{-9}

Table 12.2 Magnetic Susceptibilities *Note: Unless otherwise specified, values given are for room temperature.

Paramagnetic Materials	χ	Diamagnetic Materials	χ
Air (1 atm)	3.6×10^{-7}	Water	-9.1×10^{-6}

Table 12.2 Magnetic Susceptibilities *Note: Unless otherwise specified, values given are for room temperature.

Diamagnetic Materials

A magnetic field always induces a magnetic dipole in an atom. This induced dipole points opposite to the applied field, so its magnetic field is also directed opposite to the applied field. In paramagnetic and ferromagnetic materials, the induced magnetic dipole is masked by much stronger permanent magnetic dipoles of the atoms. However, in diamagnetic materials, whose atoms have no permanent magnetic dipole moments, the effect of the induced dipole is observable.

We can now describe the magnetic effects of diamagnetic materials with the same model developed for paramagnetic materials. In this case, however, the fictitious surface current flows opposite to the solenoid current, and the magnetic susceptibility χ is negative. Values of χ for some diamagnetic materials are also given in **Table 12.2**.

Water is a common diamagnetic material. Animals are mostly composed of water. Experiments have been performed on **frogs (https://openstaxcollege.org/l/21frogs)** and **mice (https://openstaxcollege.org/l/21mice)** in diverging magnetic fields. The water molecules are repelled from the applied magnetic field against gravity until the animal reaches an equilibrium. The result is that the animal is levitated by the magnetic field.

Ferromagnetic Materials

Common magnets are made of a ferromagnetic material such as iron or one of its alloys. Experiments reveal that a ferromagnetic material consists of tiny regions known as **magnetic domains**. Their volumes typically range from 10^{-12} to $10^{-8}\,\text{m}^3$, and they contain about 10^{17} to 10^{21} atoms. Within a domain, the magnetic dipoles are rigidly aligned in the same direction by coupling among the atoms. This coupling, which is due to quantum mechanical effects, is so strong that even thermal agitation at room temperature cannot break it. The result is that each domain has a net dipole moment. Some materials have weaker coupling and are ferromagnetic only at lower temperatures.

If the domains in a ferromagnetic sample are randomly oriented, as shown in **Figure 12.24**, the sample has no net magnetic dipole moment and is said to be unmagnetized. Suppose that we fill the volume of a solenoid with an unmagnetized ferromagnetic sample. When the magnetic field $\vec{\mathbf{B}}_0$ of the solenoid is turned on, the dipole moments of the domains rotate so that they align somewhat with the field, as depicted in **Figure 12.24**. In addition, the aligned domains tend to increase in size at the expense of unaligned ones. The net effect of these two processes is the creation of a net magnetic dipole moment for the ferromagnet that is directed along the applied magnetic field. This net magnetic dipole moment is much larger than that of a paramagnetic sample, and the domains, with their large numbers of atoms, do not become misaligned by thermal agitation. Consequently, the field due to the alignment of the domains is quite large.

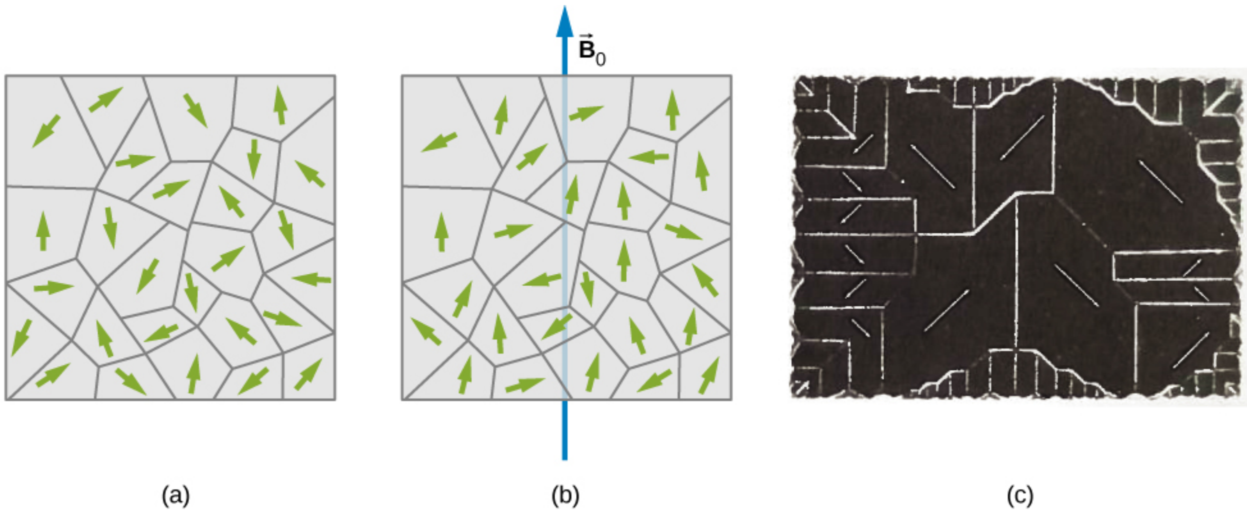


Figure 12.24 (a) Domains are randomly oriented in an unmagnetized ferromagnetic sample such as iron. The arrows represent the orientations of the magnetic dipoles within the domains. (b) In an applied magnetic field, the domains align somewhat with the field. (c) The domains of a single crystal of nickel. The white lines show the boundaries of the domains. These lines are produced by iron oxide powder sprinkled on the crystal.

Besides iron, only four elements contain the magnetic domains needed to exhibit ferromagnetic behavior: cobalt, nickel, gadolinium, and dysprosium. Many alloys of these elements are also ferromagnetic. Ferromagnetic materials can be described using **Equation 12.34** through **Equation 12.39**, the paramagnetic equations. However, the value of χ for ferromagnetic material is usually on the order of 10^3 to 10^4, and it also depends on the history of the magnetic field to which the material has been subject. A typical plot of *B* (the total field in the material) versus B_0 (the applied field) for an initially unmagnetized piece of iron is shown in **Figure 12.25**. Some sample numbers are (1) for $B_0 = 1.0 \times 10^{-4}\,\text{T}, B = 0.60\,\text{T},$ and $\chi = \left(\frac{0.60}{1.0 \times 10^{-4}}\right) - 1 \approx 6.0 \times 10^3$; (2) for $B_0 = 6.0 \times 10^{-4}\,\text{T}, B = 1.5\,\text{T},$ and $\chi = \left(\frac{1.5}{6.0 \times 10^{-4}}\right) - 1 \approx 2.5 \times 10^3$.

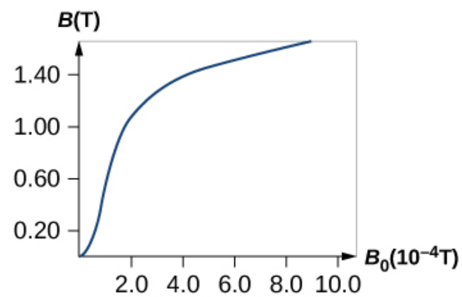


Figure 12.25 (a) The magnetic field *B* in annealed iron as a function of the applied field B_0.

When B_0 is varied over a range of positive and negative values, *B* is found to behave as shown in **Figure 12.26**. Note that the same B_0 (corresponding to the same current in the solenoid) can produce different values of *B* in the material. The magnetic field *B* produced in a ferromagnetic material by an applied field B_0 depends on the magnetic history of the material. This effect is called **hysteresis**, and the curve of **Figure 12.26** is called a hysteresis loop. Notice that *B* does not disappear when $B_0 = 0$ (i.e., when the current in the solenoid is turned off). The iron stays magnetized, which means that it has become a permanent magnet.

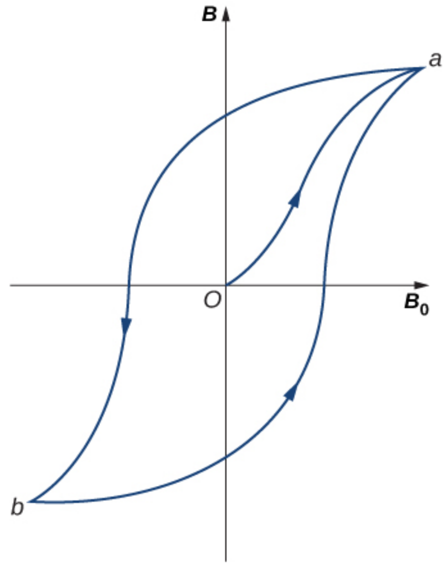


Figure 12.26 A typical hysteresis loop for a ferromagnet. When the material is first magnetized, it follows a curve from 0 to *a*. When B_0 is reversed, it takes the path shown from *a* to *b*. If B_0 is reversed again, the material follows the curve from *b* to *a*.

Like the paramagnetic sample of **Figure 12.23**, the partial alignment of the domains in a ferromagnet is equivalent to a current flowing around the surface. A bar magnet can therefore be pictured as a tightly wound solenoid with a large current circulating through its coils (the surface current). You can see in **Figure 12.27** that this model fits quite well. The fields of the bar magnet and the finite solenoid are strikingly similar. The figure also shows how the poles of the bar magnet are identified. To form closed loops, the field lines outside the magnet leave the north (N) pole and enter the south (S) pole, whereas inside the magnet, they leave S and enter N.

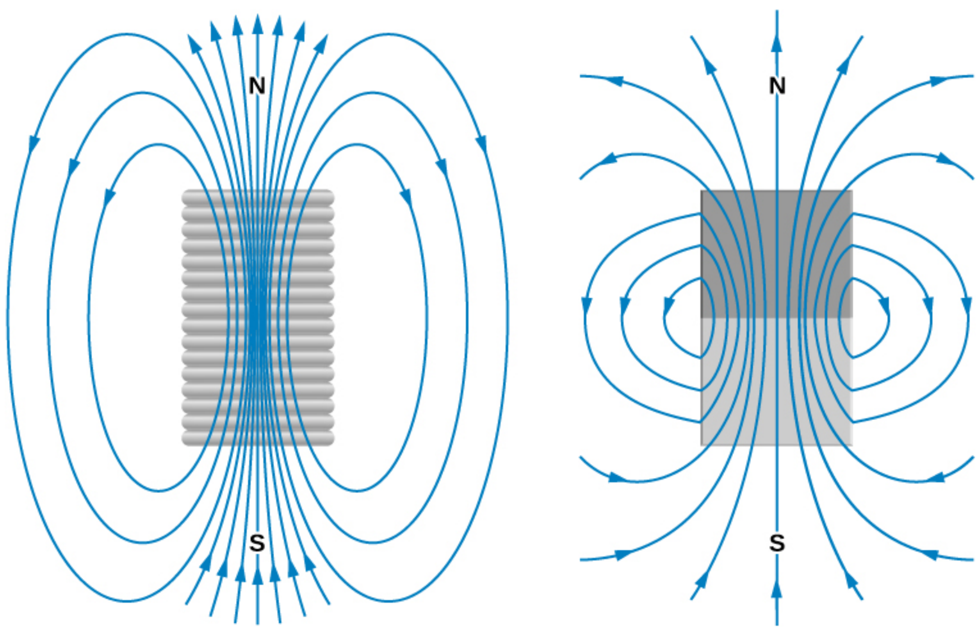


Figure 12.27 Comparison of the magnetic fields of a finite solenoid and a bar magnet.

Ferromagnetic materials are found in computer hard disk drives and permanent data storage devices (**Figure 12.28**). A material used in your hard disk drives is called a spin valve, which has alternating layers of ferromagnetic (aligning with the external magnetic field) and antiferromagnetic (each atom is aligned opposite to the next) metals. It was observed that a significant change in resistance was discovered based on whether an applied magnetic field was on the spin valve or not. This large change in resistance creates a quick and consistent way for recording or reading information by an applied current.

Figure 12.28 The inside of a hard disk drive. The silver disk contains the information, whereas the thin stylus on top of the disk reads and writes information to the disk.

Example 12.10

Iron Core in a Coil

A long coil is tightly wound around an iron cylinder whose magnetization curve is shown in **Figure 12.25**. (a) If $n = 20$ turns per centimeter, what is the applied field B_0 when $I_0 = 0.20\text{ A}$? (b) What is the net magnetic field for this same current? (c) What is the magnetic susceptibility in this case?

Strategy

(a) The magnetic field of a solenoid is calculated using **Equation 12.28**. (b) The graph is read to determine the net magnetic field for this same current. (c) The magnetic susceptibility is calculated using **Equation 12.37**.

Solution

a. The applied field B_0 of the coil is

$$\begin{aligned} B_0 &= \mu_0 n I_0 = (4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(2000/\text{m})(0.20\text{ A}) \\ B_0 &= 5.0 \times 10^{-4}\,\text{T}. \end{aligned}$$

b. From inspection of the magnetization curve of **Figure 12.25**, we see that, for this value of B_0, $B = 1.4\text{ T}$. Notice that the internal field of the aligned atoms is much larger than the externally applied field.

c. The magnetic susceptibility is calculated to be

$$\chi = \frac{B}{B_0} - 1 = \frac{1.4\text{ T}}{5.0 \times 10^{-4}\,\text{T}} - 1 = 2.8 \times 10^3.$$

Significance

Ferromagnetic materials have susceptibilities in the range of 10^3 which compares well to our results here. Paramagnetic materials have fractional susceptibilities, so their applied field of the coil is much greater than the magnetic field generated by the material.

12.8 Check Your Understanding Repeat the calculations from the previous example for $I_0 = 0.040\ \text{A}$.

CHAPTER 12 REVIEW

KEY TERMS

Ampère's law physical law that states that the line integral of the magnetic field around an electric current is proportional to the current

Biot-Savart law an equation giving the magnetic field at a point produced by a current-carrying wire

diamagnetic materials their magnetic dipoles align oppositely to an applied magnetic field; when the field is removed, the material is unmagnetized

ferromagnetic materials contain groups of dipoles, called domains, that align with the applied magnetic field; when this field is removed, the material is still magnetized

hysteresis property of ferromagnets that is seen when a material's magnetic field is examined versus the applied magnetic field; a loop is created resulting from sweeping the applied field forward and reverse

magnetic domains groups of magnetic dipoles that are all aligned in the same direction and are coupled together quantum mechanically

magnetic susceptibility ratio of the magnetic field in the material over the applied field at that time; positive susceptibilities are either paramagnetic or ferromagnetic (aligned with the field) and negative susceptibilities are diamagnetic (aligned oppositely with the field)

paramagnetic materials their magnetic dipoles align partially in the same direction as the applied magnetic field; when this field is removed, the material is unmagnetized

permeability of free space μ_0, measure of the ability of a material, in this case free space, to support a magnetic field

solenoid thin wire wound into a coil that produces a magnetic field when an electric current is passed through it

toroid donut-shaped coil closely wound around that is one continuous wire

KEY EQUATIONS

Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A}$
Contribution to magnetic field from a current element	$dB = \frac{\mu_0}{4\pi} \frac{I\,dl \sin\theta}{r^2}$
Biot–Savart law	$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}$
Magnetic field due to a long straight wire	$B = \frac{\mu_0 I}{2\pi R}$
Force between two parallel currents	$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$
Magnetic field of a current loop	$B = \frac{\mu_0 I}{2R}$ (at center of loop)
Ampère's law	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I$
Magnetic field strength inside a solenoid	$B = \mu_0 nI$
Magnetic field strength inside a toroid	$B = \frac{\mu_o NI}{2\pi r}$

Magnetic permeability	$\mu = (1 + \chi)\mu_0$
Magnetic field of a solenoid filled with paramagnetic material	$B = \mu n I$

SUMMARY

12.1 The Biot-Savart Law

- The magnetic field created by a current-carrying wire is found by the Biot-Savart law.
- The current element $Id\vec{\mathbf{l}}$ produces a magnetic field a distance *r* away.

12.2 Magnetic Field Due to a Thin Straight Wire

- The strength of the magnetic field created by current in a long straight wire is given by $B = \frac{\mu_0 I}{2\pi R}$ (long straight wire) where *I* is the current, *R* is the shortest distance to the wire, and the constant $\mu_0 = 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/s}$ is the permeability of free space.
- The direction of the magnetic field created by a long straight wire is given by right-hand rule 2 (RHR-2): Point the thumb of the right hand in the direction of current, and the fingers curl in the direction of the magnetic field loops created by it.

12.3 Magnetic Force between Two Parallel Currents

- The force between two parallel currents I_1 and I_2, separated by a distance *r*, has a magnitude per unit length given by $\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$.
- The force is attractive if the currents are in the same direction, repulsive if they are in opposite directions.

12.4 Magnetic Field of a Current Loop

- The magnetic field strength at the center of a circular loop is given by $B = \frac{\mu_0 I}{2R}$ (at center of loop), where *R* is the radius of the loop. RHR-2 gives the direction of the field about the loop.

12.5 Ampère's Law

- The magnetic field created by current following any path is the sum (or integral) of the fields due to segments along the path (magnitude and direction as for a straight wire), resulting in a general relationship between current and field known as Ampère's law.
- Ampère's law can be used to determine the magnetic field from a thin wire or thick wire by a geometrically convenient path of integration. The results are consistent with the Biot-Savart law.

12.6 Solenoids and Toroids

- The magnetic field strength inside a solenoid is

$$B = \mu_0 nI \quad \text{(inside a solenoid)}$$

where *n* is the number of loops per unit length of the solenoid. The field inside is very uniform in magnitude and direction.

- The magnetic field strength inside a toroid is

$$B = \frac{\mu_o NI}{2\pi r} \quad \text{(within the toroid)}$$

where N is the number of windings. The field inside a toroid is not uniform and varies with the distance as $1/r$.

12.7 Magnetism in Matter

- Materials are classified as paramagnetic, diamagnetic, or ferromagnetic, depending on how they behave in an applied magnetic field.
- Paramagnetic materials have partial alignment of their magnetic dipoles with an applied magnetic field. This is a positive magnetic susceptibility. Only a surface current remains, creating a solenoid-like magnetic field.
- Diamagnetic materials exhibit induced dipoles opposite to an applied magnetic field. This is a negative magnetic susceptibility.
- Ferromagnetic materials have groups of dipoles, called domains, which align with the applied magnetic field. However, when the field is removed, the ferromagnetic material remains magnetized, unlike paramagnetic materials. This magnetization of the material versus the applied field effect is called hysteresis.

CONCEPTUAL QUESTIONS

12.1 The Biot-Savart Law

1. For calculating magnetic fields, what are the advantages and disadvantages of the Biot-Savart law?

2. Describe the magnetic field due to the current in two wires connected to the two terminals of a source of emf and twisted tightly around each other.

3. How can you decide if a wire is infinite?

4. Identical currents are carried in two circular loops; however, one loop has twice the diameter as the other loop. Compare the magnetic fields created by the loops at the center of each loop.

12.2 Magnetic Field Due to a Thin Straight Wire

5. How would you orient two long, straight, current-carrying wires so that there is no net magnetic force between them? (*Hint*: What orientation would lead to one wire not experiencing a magnetic field from the other?)

12.3 Magnetic Force between Two Parallel Currents

6. Compare and contrast the electric field of an infinite line of charge and the magnetic field of an infinite line of current.

7. Is $\vec{\mathbf{B}}$ constant in magnitude for points that lie on a magnetic field line?

12.4 Magnetic Field of a Current Loop

8. Is the magnetic field of a current loop uniform?

9. What happens to the length of a suspended spring when a current passes through it?

10. Two concentric circular wires with different diameters carry currents in the same direction. Describe the force on the inner wire.

12.5 Ampère's Law

11. Is Ampère's law valid for all closed paths? Why isn't it normally useful for calculating a magnetic field?

12.6 Solenoids and Toroids

12. Is the magnetic field inside a toroid completely uniform? Almost uniform?

13. Explain why $\vec{\mathbf{B}} = 0$ inside a long, hollow copper pipe that is carrying an electric current parallel to the axis. Is $\vec{\mathbf{B}} = 0$ outside the pipe?

12.7 Magnetism in Matter

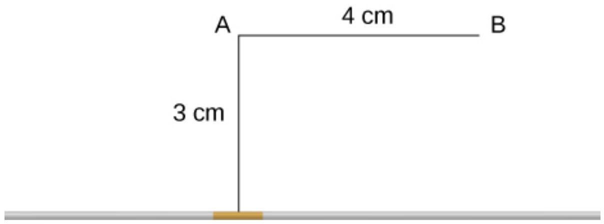
14. A diamagnetic material is brought close to a permanent magnet. What happens to the material?

15. If you cut a bar magnet into two pieces, will you end up with one magnet with an isolated north pole and another magnet with an isolated south pole? Explain your answer.

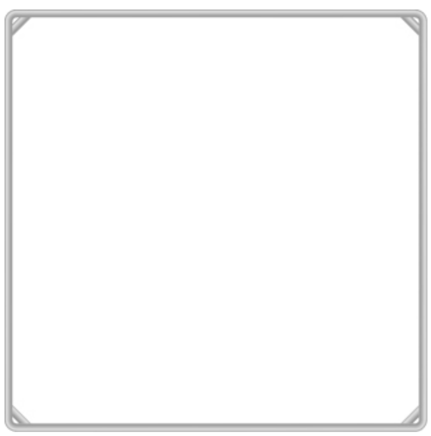
PROBLEMS

12.1 The Biot-Savart Law

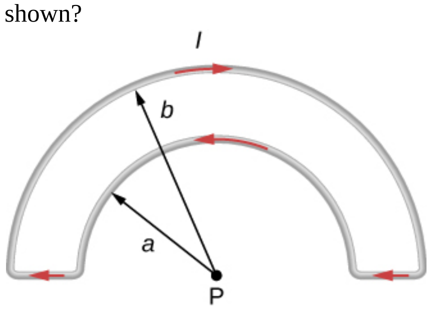
16. A 10-A current flows through the wire shown. What is the magnitude of the magnetic field due to a 0.5-mm segment of wire as measured at (a) point A and (b) point B?



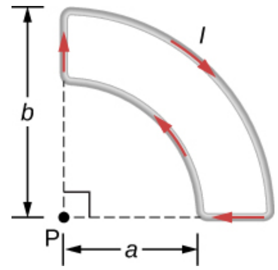
17. Ten amps flow through a square loop where each side is 20 cm in length. At each corner of the loop is a 0.01-cm segment that connects the longer wires as shown. Calculate the magnitude of the magnetic field at the center of the loop.



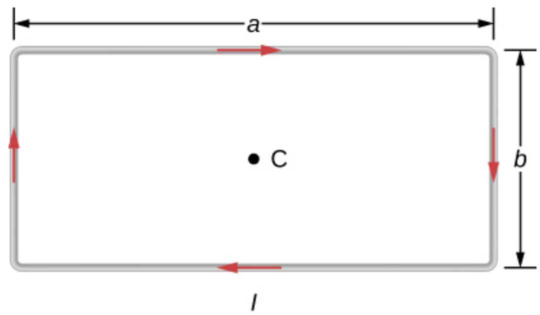
18. What is the magnetic field at P due to the current I in the wire shown?



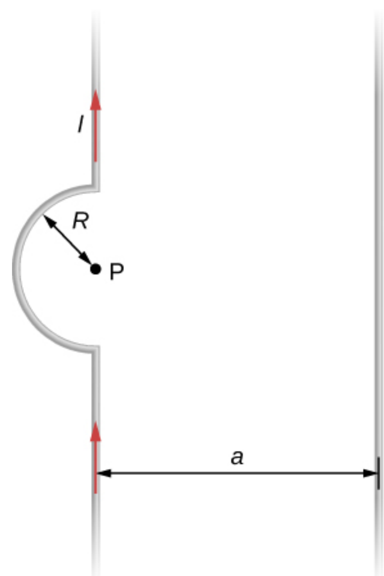
19. The accompanying figure shows a current loop consisting of two concentric circular arcs and two perpendicular radial lines. Determine the magnetic field at point P.



20. Find the magnetic field at the center C of the rectangular loop of wire shown in the accompanying figure.



21. Two long wires, one of which has a semicircular bend of radius R, are positioned as shown in the accompanying figure. If both wires carry a current I, how far apart must their parallel sections be so that the net magnetic field at P is zero? Does the current in the straight wire flow up or down?



12.2 Magnetic Field Due to a Thin Straight Wire

22. A typical current in a lightning bolt is 10^4 A. Estimate the magnetic field 1 m from the bolt.

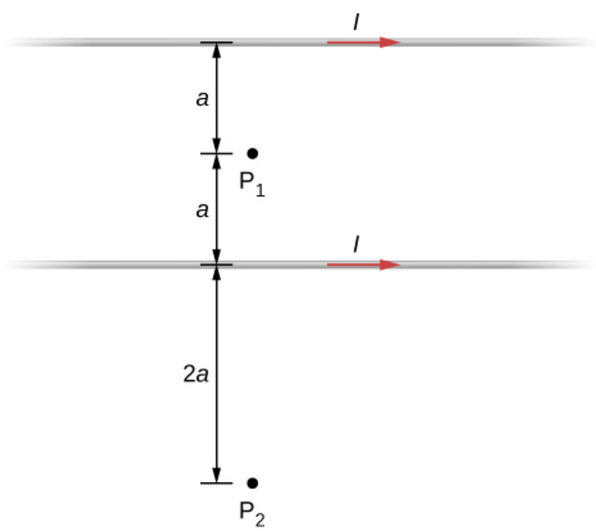
23. The magnitude of the magnetic field 50 cm from a long, thin, straight wire is $8.0\,\mu\text{T}$. What is the current through the long wire?

24. A transmission line strung 7.0 m above the ground carries a current of 500 A. What is the magnetic field on the ground directly below the wire? Compare your answer with the magnetic field of Earth.

25. A long, straight, horizontal wire carries a left-to-right current of 20 A. If the wire is placed in a uniform magnetic field of magnitude $4.0 \times 10^{-5}\,\text{T}$ that is directed vertically downward, what is the resultant magnitude of the magnetic field 20 cm above the wire? 20 cm below the wire?

26. The two long, parallel wires shown in the accompanying figure carry currents in the same direction. If $I_1 = 10\text{ A}$ and $I_2 = 20\text{ A}$, what is the magnetic field at point P?

27. The accompanying figure shows two long, straight, horizontal wires that are parallel and a distance 2*a* apart. If both wires carry current *I* in the same direction, (a) what is the magnetic field at P_1? (b) P_2?



28. Repeat the calculations of the preceding problem with the direction of the current in the lower wire reversed.

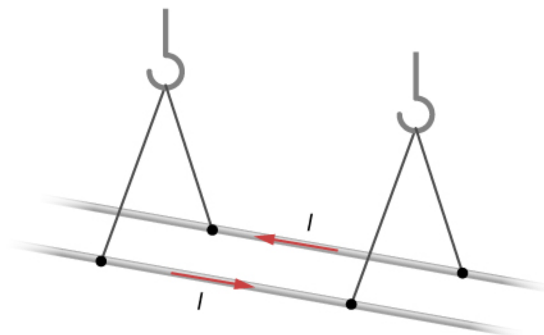
29. Consider the area between the wires of the preceding problem. At what distance from the top wire is the net magnetic field a minimum? Assume that the currents are equal and flow in opposite directions.

12.3 Magnetic Force between Two Parallel Currents

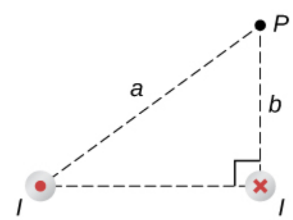
30. Two long, straight wires are parallel and 25 cm apart. (a) If each wire carries a current of 50 A in the same direction, what is the magnetic force per meter exerted on each wire? (b) Does the force pull the wires together or push them apart? (c) What happens if the currents flow in opposite directions?

31. Two long, straight wires are parallel and 10 cm apart. One carries a current of 2.0 A, the other a current of 5.0 A. (a) If the two currents flow in opposite directions, what is the magnitude and direction of the force per unit length of one wire on the other? (b) What is the magnitude and direction of the force per unit length if the currents flow in the same direction?

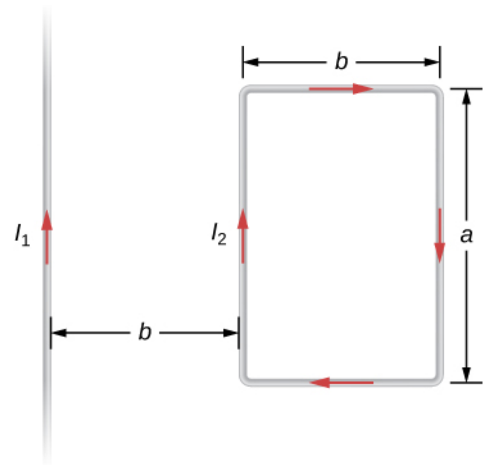
32. Two long, parallel wires are hung by cords of length 5.0 cm, as shown in the accompanying figure. Each wire has a mass per unit length of 30 g/m, and they carry the same current in opposite directions. What is the current if the cords hang at $6.0°$ with respect to the vertical?



33. A circuit with current *I* has two long parallel wire sections that carry current in opposite directions. Find magnetic field at a point *P* near these wires that is a distance *a* from one wire and *b* from the other wire as shown in the figure.



34. The infinite, straight wire shown in the accompanying figure carries a current I_1. The rectangular loop, whose long sides are parallel to the wire, carries a current I_2. What are the magnitude and direction of the force on the rectangular loop due to the magnetic field of the wire?



12.4 Magnetic Field of a Current Loop

35. When the current through a circular loop is 6.0 A, the magnetic field at its center is $2.0 \times 10^{-4}\,\text{T}$. What is the radius of the loop?

36. How many turns must be wound on a flat, circular coil of radius 20 cm in order to produce a magnetic field of magnitude $4.0 \times 10^{-5}\,\text{T}$ at the center of the coil when the current through it is 0.85 A?

37. A flat, circular loop has 20 turns. The radius of the loop is 10.0 cm and the current through the wire is 0.50 A. Determine the magnitude of the magnetic field at the center of the loop.

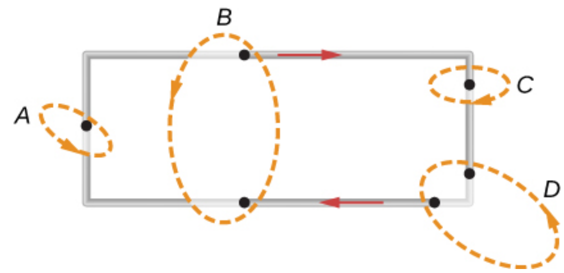
38. A circular loop of radius *R* carries a current *I*. At what distance along the axis of the loop is the magnetic field one-half its value at the center of the loop?

39. Two flat, circular coils, each with a radius *R* and wound with *N* turns, are mounted along the same axis so that they are parallel a distance *d* apart. What is the magnetic field at the midpoint of the common axis if a current *I* flows in the same direction through each coil?

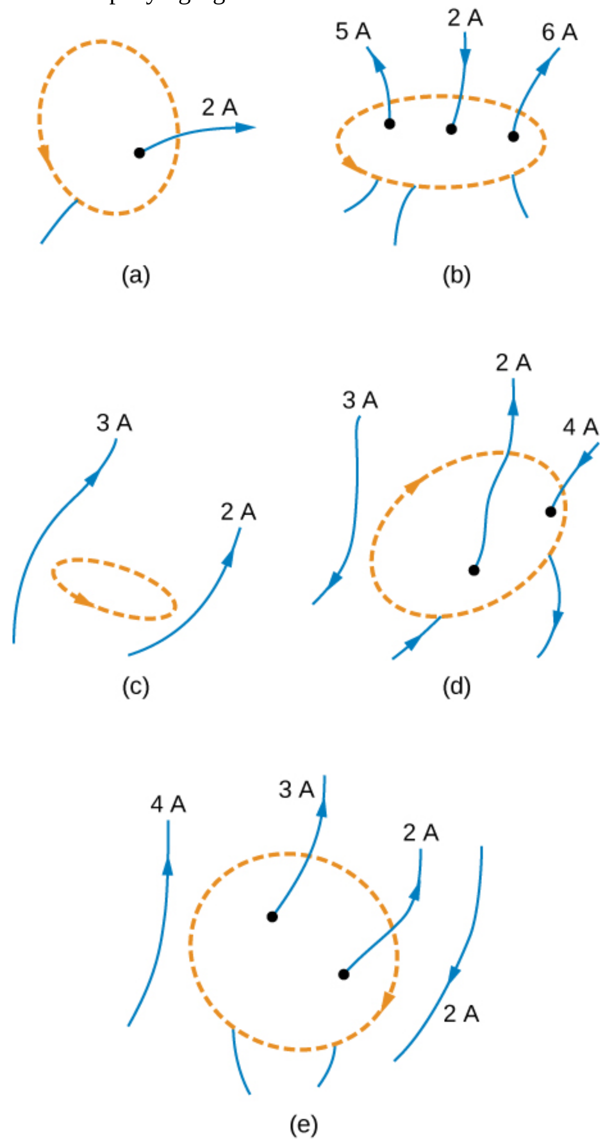
40. For the coils in the preceding problem, what is the magnetic field at the center of either coil?

12.5 Ampère's Law

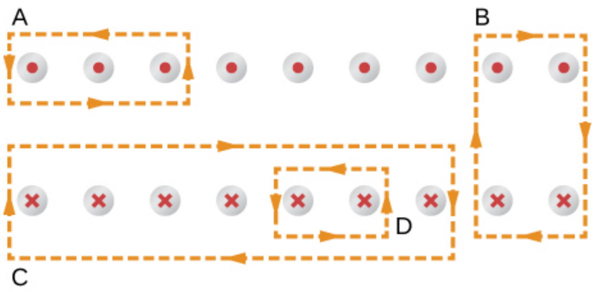
41. A current *I* flows around the rectangular loop shown in the accompanying figure. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the paths *A*, *B*, *C*, and *D*.



42. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for each of the cases shown in the accompanying figure.



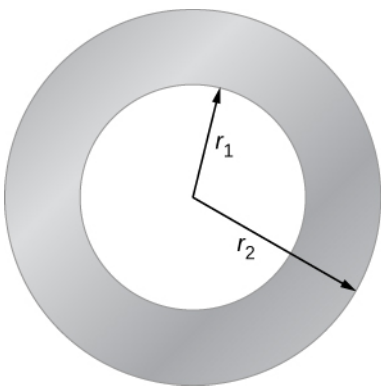
43. The coil whose lengthwise cross section is shown in the accompanying figure carries a current *I* and has *N* evenly spaced turns distributed along the length l. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the paths indicated.



44. A superconducting wire of diameter 0.25 cm carries a current of 1000 A. What is the magnetic field just outside the wire?

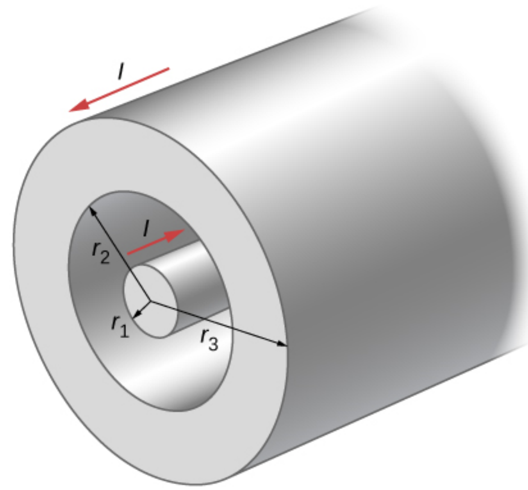
45. A long, straight wire of radius *R* carries a current *I* that is distributed uniformly over the cross-section of the wire. At what distance from the axis of the wire is the magnitude of the magnetic field a maximum?

46. The accompanying figure shows a cross-section of a long, hollow, cylindrical conductor of inner radius $r_1 = 3.0\text{ cm}$ and outer radius $r_2 = 5.0\text{ cm}$. A 50-A current distributed uniformly over the cross-section flows into the page. Calculate the magnetic field at $r = 2.0\text{ cm},$ $r = 4.0\text{ cm},$ and $r = 6.0\text{ cm}$.



47. A long, solid, cylindrical conductor of radius 3.0 cm carries a current of 50 A distributed uniformly over its cross-section. Plot the magnetic field as a function of the radial distance *r* from the center of the conductor.

48. A portion of a long, cylindrical coaxial cable is shown in the accompanying figure. A current *I* flows down the center conductor, and this current is returned in the outer conductor. Determine the magnetic field in the regions (a) $r \le r_1,$ (b) $r_2 \ge r \ge r_1,$ (c) $r_3 \ge r \ge r_2,$ and (d) $r \ge r_3$. Assume that the current is distributed uniformly over the cross sections of the two parts of the cable.



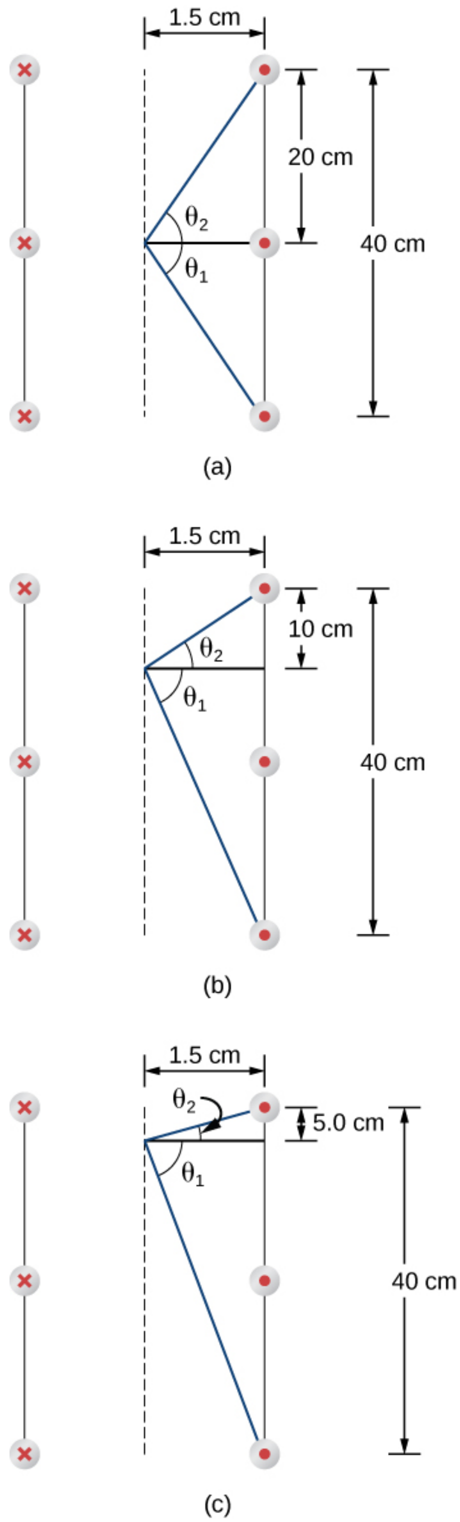
12.6 Solenoids and Toroids

49. A solenoid is wound with 2000 turns per meter. When the current is 5.2 A, what is the magnetic field within the solenoid?

50. A solenoid has 12 turns per centimeter. What current will produce a magnetic field of $2.0 \times 10^{-2}\text{ T}$ within the solenoid?

51. If a current is 2.0 A, how many turns per centimeter must be wound on a solenoid in order to produce a magnetic field of $2.0 \times 10^{-3}\text{ T}$ within it?

52. A solenoid is 40 cm long, has a diameter of 3.0 cm, and is wound with 500 turns. If the current through the windings is 4.0 A, what is the magnetic field at a point on the axis of the solenoid that is (a) at the center of the solenoid, (b) 10.0 cm from one end of the solenoid, and (c) 5.0 cm from one end of the solenoid? (d) Compare these answers with the infinite-solenoid case.



53. Determine the magnetic field on the central axis at the opening of a semi-infinite solenoid. (That is, take the opening to be at $x = 0$ and the other end to be at $x = \infty$.)

54. By how much is the approximation $B = \mu_0 nI$ in error at the center of a solenoid that is 15.0 cm long, has a diameter of 4.0 cm, is wrapped with *n* turns per meter, and carries a current *I*?

55. A solenoid with 25 turns per centimeter carries a current *I*. An electron moves within the solenoid in a circle that has a radius of 2.0 cm and is perpendicular to the axis of the solenoid. If the speed of the electron is 2.0×10^5 m/s, what is *I*?

56. A toroid has 250 turns of wire and carries a current of 20 A. Its inner and outer radii are 8.0 and 9.0 cm. What are the values of its magnetic field at $r = 8.1$, 8.5, and 8.9 cm?

57. A toroid with a square cross section 3.0 cm × 3.0 cm has an inner radius of 25.0 cm. It is wound with 500 turns of wire, and it carries a current of 2.0 A. What is the strength of the magnetic field at the center of the square cross section?

12.7 Magnetism in Matter

58. The magnetic field in the core of an air-filled solenoid is 1.50 T. By how much will this magnetic field decrease if the air is pumped out of the core while the current is held constant?

59. A solenoid has a ferromagnetic core, *n* = 1000 turns per meter, and *I* = 5.0 A. If *B* inside the solenoid is 2.0 T, what is χ for the core material?

60. A 20-A current flows through a solenoid with 2000 turns per meter. What is the magnetic field inside the solenoid if its core is (a) a vacuum and (b) filled with liquid oxygen at 90 K?

61. The magnetic dipole moment of the iron atom is about $2.1 \times 10^{-23}\,\text{A} \cdot \text{m}^2$. (a) Calculate the maximum magnetic dipole moment of a domain consisting of 10^{19} iron atoms. (b) What current would have to flow through a single circular loop of wire of diameter 1.0 cm to produce this magnetic dipole moment?

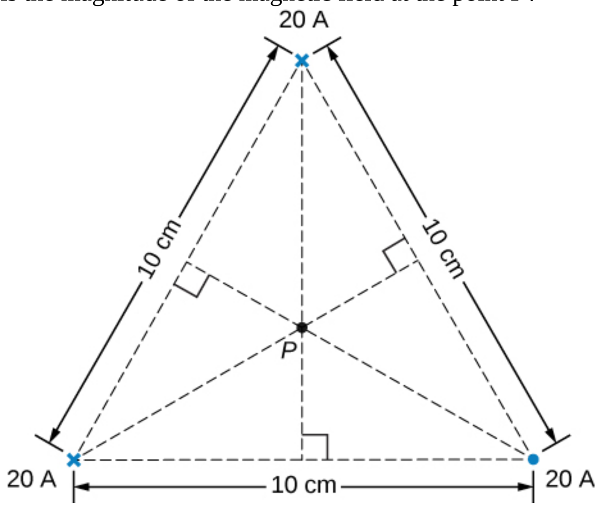
62. Suppose you wish to produce a 1.2-T magnetic field in a toroid with an iron core for which $\chi = 4.0 \times 10^3$. The toroid has a mean radius of 15 cm and is wound with 500 turns. What current is required?

63. A current of 1.5 A flows through the windings of a large, thin toroid with 200 turns per meter. If the toroid is filled with iron for which $\chi = 3.0 \times 10^3$, what is the magnetic field within it?

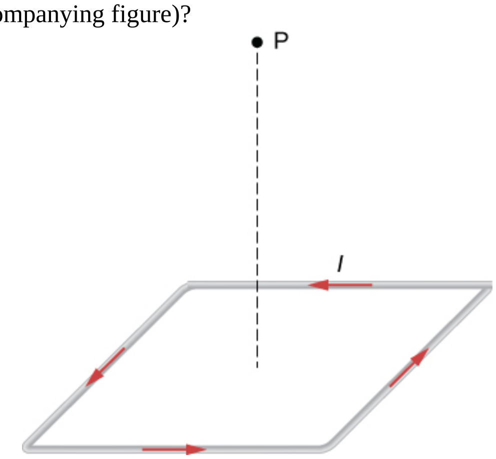
64. A solenoid with an iron core is 25 cm long and is wrapped with 100 turns of wire. When the current through the solenoid is 10 A, the magnetic field inside it is 2.0 T. For this current, what is the permeability of the iron? If the current is turned off and then restored to 10 A, will the magnetic field necessarily return to 2.0 T?

ADDITIONAL PROBLEMS

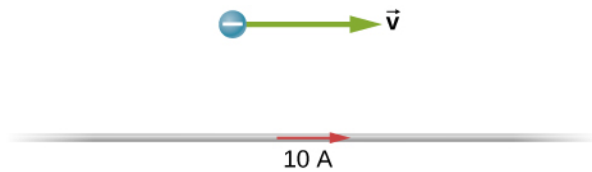
65. Three long, straight, parallel wires, all carrying 20 A, are positioned as shown in the accompanying figure. What is the magnitude of the magnetic field at the point *P*?



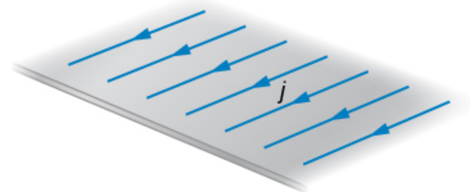
66. A current *I* flows around a wire bent into the shape of a square of side *a*. What is the magnetic field at the point P that is a distance *z* above the center of the square (see the accompanying figure)?



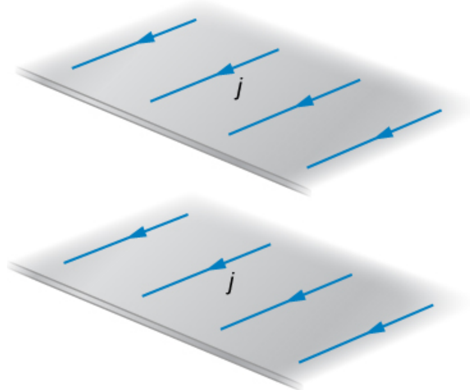
67. The accompanying figure shows a long, straight wire carrying a current of 10 A. What is the magnetic force on an electron at the instant it is 20 cm from the wire, traveling parallel to the wire with a speed of 2.0×10^5 m/s? Describe qualitatively the subsequent motion of the electron.



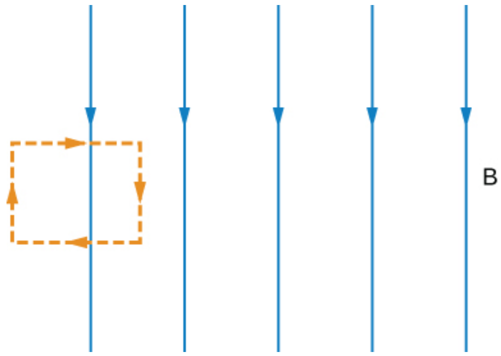
68. Current flows along a thin, infinite sheet as shown in the accompanying figure. The current per unit length along the sheet is *J* in amperes per meter. (a) Use the Biot-Savart law to show that $B = \mu_0 J / 2$ on either side of the sheet. What is the direction of $\vec{\mathbf{B}}$ on each side? (b) Now use Ampère's law to calculate the field.



69. (a) Use the result of the previous problem to calculate the magnetic field between, above, and below the pair of infinite sheets shown in the accompanying figure. (b) Repeat your calculations if the direction of the current in the lower sheet is reversed.



70. We often assume that the magnetic field is uniform in a region and zero everywhere else. Show that in reality it is impossible for a magnetic field to drop abruptly to zero, as illustrated in the accompanying figure. (*Hint*: Apply Ampère's law over the path shown.)



71. How is the percentage change in the strength of the magnetic field across the face of the toroid related to the percentage change in the radial distance from the axis of the toroid?

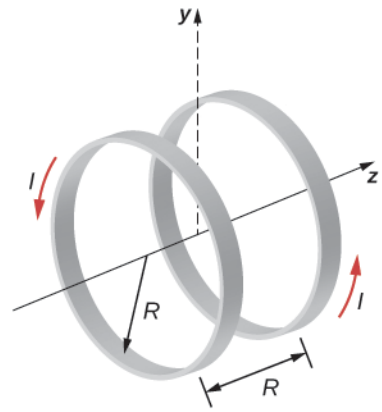
72. Show that the expression for the magnetic field of a toroid reduces to that for the field of an infinite solenoid in the limit that the central radius goes to infinity.

73. A toroid with an inner radius of 20 cm and an outer radius of 22 cm is tightly wound with one layer of wire that has a diameter of 0.25 mm. (a) How many turns are there on the toroid? (b) If the current through the toroid windings is 2.0 A, what is the strength of the magnetic field at the center of the toroid?

74. A wire element has $d\vec{\mathbf{l}}$, $Id\vec{\mathbf{l}} = \mathbf{J}Adl = \mathbf{J}dv$, where A and dv are the cross-sectional area and volume of the element, respectively. Use this, the Biot-Savart law, and $\mathbf{J} = ne\mathbf{v}$ to show that the magnetic field of a moving point charge q is given by:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\hat{\mathbf{r}}}{r^2}$$

75. A reasonably uniform magnetic field over a limited region of space can be produced with the Helmholtz coil, which consists of two parallel coils centered on the same axis. The coils are connected so that they carry the same current I. Each coil has N turns and radius R, which is also the distance between the coils. (a) Find the magnetic field at any point on the z-axis shown in the accompanying figure. (b) Show that dB/dz and $\frac{d^2B}{dz^2}$ are both zero at $z = 0$. (These vanishing derivatives demonstrate that the magnetic field varies only slightly near $z = 0$.)

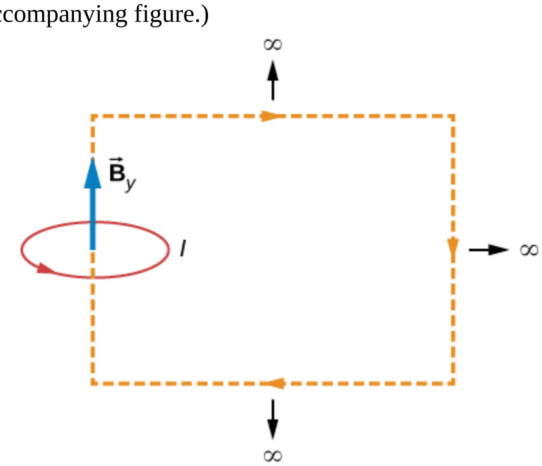


76. A charge of $4.0\,\mu\text{C}$ is distributed uniformly around a thin ring of insulating material. The ring has a radius of 0.20 m and rotates at $2.0\times10^4\,\text{rev/min}$ around the axis that passes through its center and is perpendicular to the plane of the ring. What is the magnetic field at the center of the ring?

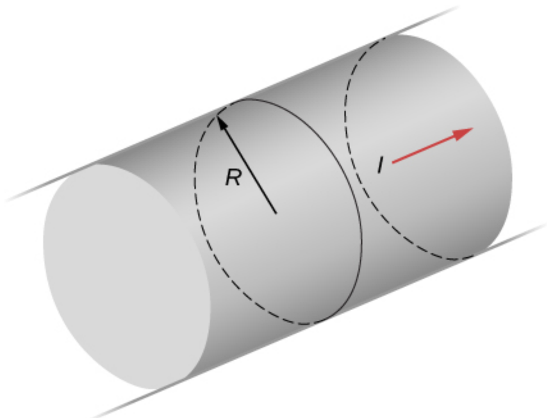
77. A thin, nonconducting disk of radius R is free to rotate around the axis that passes through its center and is perpendicular to the face of the disk. The disk is charged uniformly with a total charge q. If the disk rotates at a constant angular velocity ω, what is the magnetic field at its center?

78. Consider the disk in the previous problem. Calculate the magnetic field at a point on its central axis that is a distance y above the disk.

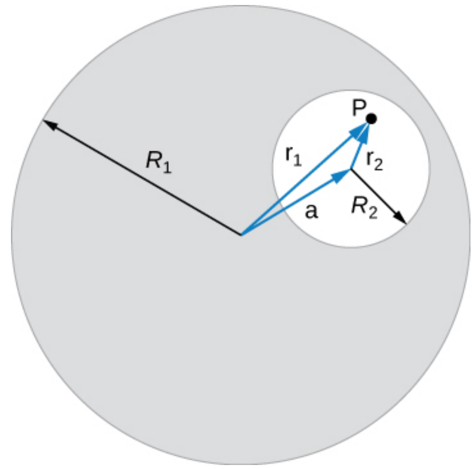
79. Consider the axial magnetic field $B_y = \mu_0 IR^2/2(y^2+R^2)^{3/2}$ of the circular current loop shown below. (a) Evaluate $\int_{-a}^{a} B_y\,dy$. Also show that $\lim_{a\to\infty}\int_{-a}^{a} B_y dy = \mu_0 I$. (b) Can you deduce this limit without evaluating the integral? (*Hint:* See the accompanying figure.)



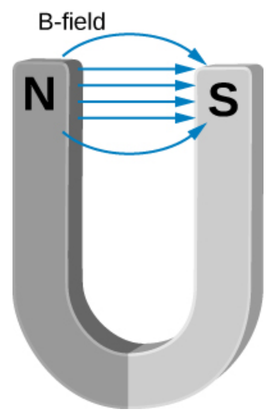
80. The current density in the long, cylindrical wire shown in the accompanying figure varies with distance r from the center of the wire according to $J = cr$, where c is a constant. (a) What is the current through the wire? (b) What is the magnetic field produced by this current for $r \leq R$? For $r \geq R$?



81. A long, straight, cylindrical conductor contains a cylindrical cavity whose axis is displaced by a from the axis of the conductor, as shown in the accompanying figure. The current density in the conductor is given by $\vec{\mathbf{J}} = J_0\hat{\mathbf{k}}$, where J_0 is a constant and $\hat{\mathbf{k}}$ is along the axis of the conductor. Calculate the magnetic field at an arbitrary point P in the cavity by superimposing the field of a solid cylindrical conductor with radius R_1 and current density $\vec{\mathbf{J}}$ onto the field of a solid cylindrical conductor with radius R_2 and current density $-\vec{\mathbf{J}}$. Then use the fact that the appropriate azimuthal unit vectors can be expressed as $\hat{\theta}_1 = \hat{k} \times \hat{r}_1$ and $\hat{\theta}_2 = \hat{k} \times \hat{r}_2$ to show that everywhere inside the cavity the magnetic field is given by the constant $\vec{\mathbf{B}} = \frac{1}{2}\mu_0 J_0 \mathbf{k} \times \mathbf{a}$, where $\mathbf{a} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_1 = r_1\hat{r}_1$ is the position of P relative to the center of the conductor and $\mathbf{r}_2 = r_2\hat{r}_2$ is the position of P relative to the center of the cavity.

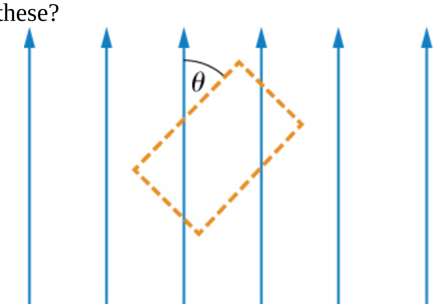


82. Between the two ends of a horseshoe magnet the field is uniform as shown in the diagram. As you move out to outside edges, the field bends. Show by Ampère's law that the field must bend and thereby the field weakens due to these bends.



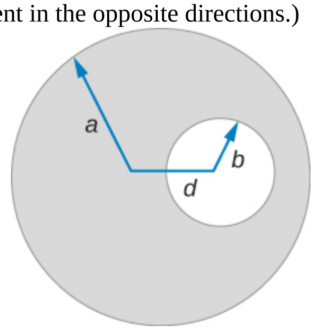
83. Show that the magnetic field of a thin wire and that of a current loop are zero if you are infinitely far away.

84. An Ampère loop is chosen as shown by dashed lines for a parallel constant magnetic field as shown by solid arrows. Calculate $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for each side of the loop then find the entire $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$. Can you think of an Ampère loop that would make the problem easier? Do those results match these?

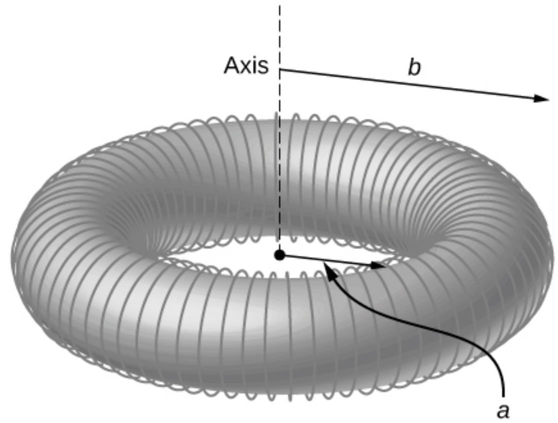


85. A very long, thick cylindrical wire of radius R carries a current density J that varies across its cross-section. The magnitude of the current density at a point a distance r from the center of the wire is given by $J = J_0 \frac{r}{R}$, where J_0 is a constant. Find the magnetic field (a) at a point outside the wire and (b) at a point inside the wire. Write your answer in terms of the net current I through the wire.

86. A very long, cylindrical wire of radius a has a circular hole of radius b in it at a distance d from the center. The wire carries a uniform current of magnitude I through it. The direction of the current in the figure is out of the paper. Find the magnetic field (a) at a point at the edge of the hole closest to the center of the thick wire, (b) at an arbitrary point inside the hole, and (c) at an arbitrary point outside the wire. (*Hint:* Think of the hole as a sum of two wires carrying current in the opposite directions.)



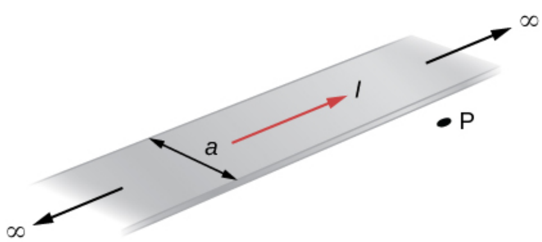
87. Magnetic field inside a torus. Consider a torus of rectangular cross-section with inner radius a and outer radius b. N turns of an insulated thin wire are wound evenly on the torus tightly all around the torus and connected to a battery producing a steady current I in the wire. Assume that the current on the top and bottom surfaces in the figure is radial, and the current on the inner and outer radii surfaces is vertical. Find the magnetic field inside the torus as a function of radial distance r from the axis.



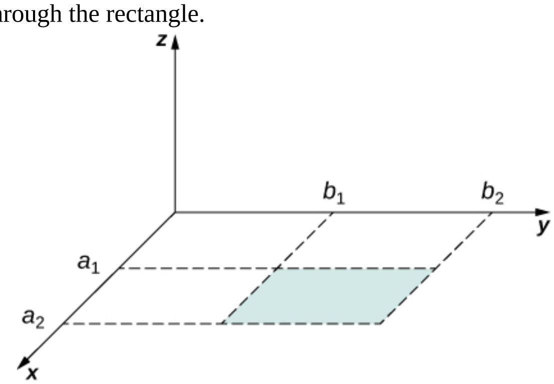
88. Two long coaxial copper tubes, each of length L, are connected to a battery of voltage V. The inner tube has inner radius a and outer radius b, and the outer tube has inner radius c and outer radius d. The tubes are then disconnected from the battery and rotated in the same direction at angular speed of ω radians per second about their common axis. Find the magnetic field (a) at a point inside the space enclosed by the inner tube $r < a$, and (b) at a point between the tubes $b < r < c$, and (c) at a point outside the tubes $r > d$. (*Hint:* Think of copper tubes as a capacitor and find the charge density based on the voltage applied, $Q = VC$, $C = \frac{2\pi\varepsilon_0 L}{\ln(c/b)}$.)

CHALLENGE PROBLEMS

89. The accompanying figure shows a flat, infinitely long sheet of width a that carries a current I uniformly distributed across it. Find the magnetic field at the point P, which is in the plane of the sheet and at a distance x from one edge. Test your result for the limit $a \to 0$.



90. A hypothetical current flowing in the z-direction creates the field $\vec{\mathbf{B}} = C\left[\left(x/y^2\right)\hat{\mathbf{i}} + (1/y)\hat{\mathbf{j}}\right]$ in the rectangular region of the xy-plane shown in the accompanying figure. Use Ampère's law to find the current through the rectangle.



91. A nonconducting hard rubber circular disk of radius R is painted with a uniform surface charge density σ. It is rotated about its axis with angular speed ω. (a) Find the magnetic field produced at a point on the axis a distance h meters from the center of the disk. (b) Find the numerical value of magnitude of the magnetic field when

$\sigma = 1\text{C/m}^2$, $R = 20$ cm, $h = 2$ cm, and $\omega = 400$ rad/sec, and compare it with the magnitude of magnetic field of Earth, which is about 1/2 Gauss.